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MMT-008 (P)

MACS (MASTER'S IN MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

Term-End Examination

August, 2011

00780

MMT-008 (P) : PRACTICAL (PROBABILITY AND STATISTICS)

Time : 1½ hours *Maximum Marks : 40*

Note : *There is **one** question in this paper worth **30** marks. Remaining **10** marks are for **viva-voce**.*

1. Write a program in 'C' language to compute T^2 for testing $H_0 : \mu = [\,7, 11\,]$, using the data matrix from a bivariate normal population : **30**

$$X = \begin{bmatrix} 2 & 12 \\ 8 & 9 \\ 6 & 9 \\ 8 & 10 \end{bmatrix}$$

Also, extend this program to specify the distribution of T^2.

No. of Printed Pages : 6 **MMT-008**

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE) M.Sc. (MACS)

00728

Term-End Examination

December, 2011

MMT-008 : PROBABILITY AND STATISTICS

Time : 3 hours *Maximum Marks : 100*

Note : *Question number 8 is* ***compulsory****. Answer any* ***six*** *questions from question number* ***1 to 7****. Use of calculator is* ***not*** *allowed.*

1. (a) Two random variables X and Y have the following joint p.d.f. **9**

$$f(x,y)=\frac{3}{16}(x^2+y^2)\,;\,0<y<x\leq 2$$

(i) Find the marginal distribution of y.

(ii) Conditional expectation of Y given $X=x$

(iii) $P\left[\frac{1}{2}<Y<1 \,\middle|\, \frac{1}{2}<X<\frac{3}{2}\right]$

(b) Explain Markov chain with the help of an example state and derive chapman-kolomogorov equation. **6**

2\. (a) A Markov chain has an initial distribution $u^{(0)} = \left\{\frac{1}{6}, \frac{1}{2}, \frac{1}{3}\right\}$ and has the following transition matrix : 7

$$P = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{bmatrix}$$

(i) Check whether this is irreducible and a periodic.

(ii) Find its stationary distribution. Is it unique ? Justify.

(iii) Verify that limiting distribution of the chain is stationary.

(b) The arrivals at a counter in a bank occur in accordance with a Poisson process at an average rate of 8 per hour. The duration of service of a customer has exponential distribution with a mean of 6 minutes. Find the probability that an arriving customer 6

(i) has to wait on arrival,

(ii) finds 4 customers in the system,

(iii) has to spend less than 15 minutes in the bank.

Also, estimate the fraction of the total time that the counter is busy.

(c) Prove that the processes $\{N_t, t \geq 0\}$ and $\{N_t + X, -1, t \geq 0\}$ have same probability law. 2

3. (a) Consider a system of two servers where customers from outside the system arrive at server 1 at a Poisson rate 4 and at server 2 at a Poisson rate 5. The service rates of servers 1 and 2 are 8 and 10 respectively. A customer on completion of service at server 1 is equally likely to go to server 2 to leave the system (i.e., $P_{11} = 0$, $P_{12} = 1/2$) whereas a departure from server 2 will go to server $\perp$ with 25 percent of the time and will depart the system otherwise (i.e. $P_{21} = 1/4$, $P_{22} = 0$). Determine the limiting probabilities, expected length of queue and expected waiting time of a customer in the system. 9

(b) Let $\{X_n\}$ be a branching process, where the probability distribution of the numbers of offspring is geometric, with $p_n = P[\xi = n] = qp^n$, $n = 0, 1, 2...$ and $q = 1 - p$. Then, find the probability generating function of $\{X_n\}$. 6

4. (a) Describe $M \mid M \mid K \mid \infty$ queueing system, stating the assumptions. The arrival of customers follows Poisson distribution and service time has an exponential distribution. Derive the expression for probability that system will be idle. 9

(b) What do you mean by spectral decomposition of a matrix A ? Obtain the spectral decomposition of 6

$$A = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

5. (a) A bag contains 3 white and 5 black balls. 4 balls are transferred to another empty bag. From this bag, now a ball is drawn and is found to be white. What is the probability that out of four balls transferred, 3 are white and 1 is black ? 5

(b) Consider the covariance matrix for random vector $[X_1, X_2]$ as given below : 10

$$\Sigma = \begin{bmatrix} 1 & 4 \\ 4 & 100 \end{bmatrix}$$

obtain the principle components using Σ. Obtain the correlation matrix R associated with Σ and show that principle components obtained from R are different from those obtained from Σ. Find the proportion of total population variance explained by the components and interpret it.

6. (a) Two samples of size 100 bars and 160 bars were taken from the lots produced by method 1 and method 2. Two characteristics X_1 = lather and X_2 = mildness were measured. The summary statistics for bars produced by methods 1 and 2 is given by 10

$$\bar{X}_1 = \begin{bmatrix} 8 \\ 4 \end{bmatrix}, \quad S_1 = \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}$$

$$\bar{X}_2 = \begin{bmatrix} 10 \\ 4 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 2 & 1 \\ 1 & 6 \end{bmatrix}$$

Test at 5% level of significance whether $\mu_1 = \mu_2$ or not.

[You may like to use the following values $F_{2,\,120} = 3.07$, $F_{2,\,\infty} = 3.00$, $F_{3,\,120} = 2.68$.]

(b) Let $y \sim N_3(\mu, \Sigma)$, where $\mu = \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix}$ and 5

$\Sigma = \begin{bmatrix} 6 & 1 & 2 \\ 1 & 8 & 4 \\ 2 & 4 & 9 \end{bmatrix}$. Obtain the distribution of

$z = cy$ where $c = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 5 & 1 \end{bmatrix}$.

7. (a) What do you mean by cannonical correlation ? Explain various steps of cannonical correlation analysis if data on N objects for two sets of variables say Set I X_1, X_2, X_3 and Set II Y_1, Y_2 are available. 8

(b) Let X be a random vector with covariance matrix . 7

$$\begin{bmatrix} 4 & 2 & 6 \\ 2 & 17 & 27 \\ 6 & 27 & 70 \end{bmatrix}$$

Find a lower triangular matrix B and its inverse such that the components of $Y = B^{-1} X$ are uncorrelated with each other having variance unity.

8. State whether the following statements are true or false. Justify your answer : **2x5=10**

(a) Any finite aperiodic irreducible chain is necessarily ergodic having stationary distribution.

(b) Let $\{\hat{N}_t : t > 0\}$ be defined as $\hat{N}_0 = 0$ and $\hat{N}_t = \text{Sup}\{k : S_k \leq t\}$, then $\hat{N}$ coincides with renewal process N.

(c) If a queueing system is represented by $E_3 | E_1 | 1$, it means the interarrival time is exponential and queue discipline is LCFS.

(d) If A is an idempotent matrix and P is any singular matrix, then $PAP^{-1} = B$ is also an idempotent matrix.

(e) Cannonical correlation is a particular case of multiple correlation.

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MMT-008 (P)

00228

MACS (MASTERS IN MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

Term-End Examination

December, 2011

MMT-008 (P) : PROBABILITY AND STATISTICS

Time : 1½ hours *Maximum Marks : 40*

Note : *There is* ***one*** *question in this paper worth* ***30*** *marks. Remaining* ***10*** *marks are for* ***viva-voce.***

1. Consider y = $(y_1, y_2, y_3)'$ having $N_3(\mu,\Sigma)$, where **30**

$$\mu = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} \text{ and } \Sigma = \begin{bmatrix} 9 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 6 \end{bmatrix}$$

write a progam in 'C' language to find the marginal distribution of y_1, y_2 and y_3. Also extend this program to find the conditional distribution of y_1 given y_2 and y_3.

No. of Printed Pages : 6 **MMT-008**

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE) M.Sc. (MACS)

Term-End Examination 00453

June, 2012

MMT-008 : PROBABILITY AND STATISTICS

Time : 3 hours *Maximum Marks : 100*

Note : *Question number 8 is* ***compulsory****. Answer any* ***six*** *questions from question number* ***1 to 7****. Use of calculator is* ***not*** *allowed.*

1. (a) Describe birth and death process. If $\lambda_k = \lambda$ and $\mu_k = k_u$, $k \geq 0$, λ, $\mu > 0$ then show that the stationary distribution of the process always exists. Obtain steady state distribution of the process. 8

(b) Using following transition matrix for a Markov chain find : 7

(i) Whether the chain is irreducible ? Why ?

(ii) Probabilities of ultimate return to the states.

(iii) Mean recurrence times of the states.

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 3/4 & 0 & 1/4 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

2. (a) Consider a branching process $\{X_n\}$. Given $X_0 = 1$ and probability distribution of number of offsprings to any individual is geometric. Find the probability generating function (p.g.f) of $\{X_n\}$. 5

(b) Find P^n and its limiting value if any for the P-matrix of a Markov chain given below. 7

$$P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

(c) Suppose that families migrate to an area at a Poisson rate $\lambda = 2$ per week. The number of people in each family is independent and takes on values 1, 2, 3, 4 with respective probabilities $\frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{6}$. Find the expected value and variance of the number of individuals migrating to this area during a fixed five week period. 3

3. (a) In a city 20% of the population were infected from TB. A diagnostic test reports positive in 95% cases when performed on TB infected person and reports positive in 15% cases when performed on non-infected person. An individual was choosen at random from the city and the test was performed. What is the probability that the result of the test was positive ? If it is positive report then what is the probability that the choosen individual was infected from TB ? 8

(b) Customers arrive in a bank according to poission law at a rate 2 per five minutes. Service time in the bank follows exponential distribution with mean 2 minutes. Find : 7

(i) probability that bank is empty of customers.

(ii) average number of customers in the bank when a customer arrives.

(iii) expected time spent by a customer in the bank.

4. (a) The joint p.d.f of two random variables X and Y is given by : 8

$$f(x,y) = \frac{9\,(x+y+1)}{2(1+x)^4\,(1+y)^4}\,; \qquad 0 \le x < \infty$$
$$0 < y < \infty$$

Find the marginal distributions of X and Y, and the conditional distribution of Y for $X = x$.

(b) In a renewal process, renewal period X_n is iid Brenoulli (p). Show that the distribution of N_t will be negative binomial. What will be the renewal function for this process ? 7

5. (a) Let the vector y be distributed as $N_3\,(\mu, \Sigma)$, 8

$$\text{where } \mu = \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix} \text{ and } \Sigma = \begin{pmatrix} 3 & -3 & -1 \\ -3 & 6 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

(b) Find a, b, c for which matrix A will be orthogonal : 7

$$A = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & a \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & b \\ 0 & \frac{1}{\sqrt{2}} & c \end{pmatrix}$$

8. State whether following statements are *true* or *false*. Justify your answer. **2x5=10**

(a) If two events A and B are non-null and mutually exclusive, then both the events are independent.

(b) Sum of elements of a 2×2 transition matrix of a Markov Chain is 4.

(c) The quadratic form $Q = 2x_1^2 - 3x_2^2 - 6x_1 x_2$ is negative definite.

(d) Principal components depend on the scales used to measure the variables.

(e) If $X \sim N_p(\mu, \Sigma)$ then the linear combinations of the components of X are normally distributed.

Find :

(i) marginal distribution of $\begin{pmatrix} y_1 \\ y_3 \end{pmatrix}$

(ii) conditional distribution of y_1 given y_2, y_3 and of $y_1\ y_2$ given y_3.

(iii) r_{12} and the partial correlation coefficient $r_{12..3}$.

(b) From the samples of sizes 80 and 100 from two populations following summary statistics were obtained. 7

$$X_1 = \begin{pmatrix} 8 \\ 4 \end{pmatrix}\ X_2 = \begin{pmatrix} 10 \\ 4 \end{pmatrix} S_1 = \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix} S_2 = \begin{pmatrix} 2 & 1 \\ 1 & 6 \end{pmatrix}$$

Where X_1 and X_2 are the means and S_1 and S_2 are the standard deviations of two populations. Test equality of population means at 5% level of significance. Assume $\Sigma_1 = \Sigma_2$ given.

You may like to use the following values.

$F_{2,\,177}^{(.05)} = 3.04$ $\quad F_{2,100}^{(0.05)} = 3.1$

$F_{2,\,80}^{(0.05)} = 3.15$

6. (a) $y \sim N_3\,(\mu, \Sigma)$ where : 8

$$\mu = \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix} \text{ and } \Sigma = \begin{pmatrix} 2 & -2 & -1 \\ -2 & 5 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

Obtain

(i) distribution of $c\,y$ where ,

$$C = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

(ii) a linear combination $Z = l'\,y$ such that $Z \sim N\,(0, 1)$.

(b) Evaluate Hotelling's T^2 for testing $H_o : \mu' = [8, 10]$ from a sample expressed by the following data matrix : **7**

$X = \begin{pmatrix} 2 & 8 & 6 & 8 \\ 12 & 9 & 9 & 10 \end{pmatrix}$ and specify the distribution of T^2.

7. (a) Variance -covariance matrix of three variables X_1, X_2 and X_3 is : **8**

$$\Sigma = \begin{pmatrix} 20 & 6 & 5 \\ 6 & 31 & 4 \\ 5 & 4 & 43 \end{pmatrix}$$

The eigen values and corresponding eigen vectors are :

$\lambda_1 = 45.9 \quad a_1^{-1} \; (0.25, 0.34, 0.90)$

$\lambda_2 = 31.1 \quad a_2^{-1} \; (-0.28, -0.87, 0.41)$

$\lambda_3 = 17.0 \quad a_3^{-1} \; (-0.92, 0.36, 0.12)$

(i) Obtain principal components.

(ii) Obtain variances of principal components

(iii) Show that total variation explained by principal components is equal to the total variances of the variables.

(iv) Obtain proportion of variations explained by Ist two components.

No. of Printed Pages : 6 **MMT-008**

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE) M.Sc. (MACS)

Term-End Examination 00641

December, 2012

MMT-008 : PROBABILITY AND STATISTICS

Time : 3 hours *Maximum Marks : 100*

(Weightage : 50%)

Note : *Question No. 8 is* ***compulsory****. Answer* ***any six*** *questions from question No.* **1** *to* **7**. *Use of calculator is* ***not*** *allowed.*

1. (a) Random variables X and Y have joint p.d.f 7

$$f(x, y) = \frac{1}{8}(y^2 - x^2)\, e^{-y}\ ;\ |X| \leq y,\ y > 0.$$

(i) Find marginal distributions of X and Y and hence prove that X and Y are dependent.

(ii) Check whether X and Y are correlated or not.

(b) Using the method of spectral decomposition evaluate p^n for the following transition matrix 8

$$P = \begin{bmatrix} 0.7 & 0.3 \\ .6 & 0.4 \end{bmatrix}.$$

Hence obtain $\lim_{n \to \infty} p^n$.

2. (a) Differentiate between irreducible and a periodic Markov Chain with the help of examples. Define recurrence and transient states. If n step transition probabilities of a given state i is $p_{ij}^{(n)} = \frac{1}{2}$ for n > No then 6

(i) Show that all the states are a periodic and non-null persistent.

(ii) Find the mean recurrence time for the state i.

(b) A system has two identical components, but uses only one at a time. If first one fails, it automatically switches on to the second. The system fails, when both the components fail. At that time both are replaced instantaneously. Assuming that the life time distribution of the component is exponential with parameter λ, express the system in a renewal process frame work. Find the inter-occurance time distribution and its laplace transform. 9

3. (a) Consider a planned replacement policy that takes place every 3 years ie. a machine required for continuous use is replaced on failure or at the end of three years. Compute. 7

(i) long term rate of replacement

(ii) long term rate of failure

(iii) long term rate of planned replacement.

Assume that successive machine life times are uniformly distributed over the interval [2, 5] years.

(b) Describe M/M/K/N queueing system, stating the assumptions. Derive the expression for probability of i customers in the system in steady state. Also, derive expression for expected number of customers in the queue. 8

4. (a) Write the assumptions for the system M/D/2/N in queueing notation. In a single server channel in doctor's clinic, patients arrive and join the queue at the end every α minutes and leave the clinic every β minutes. Assuming the queue with n patients, find the queue length ? 5

(b) Consider the mean vector $\mu_x = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $\mu_y = 9$, and the covariance matrices of x_1, x_2 and y are 10

$$\sum_{xx} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \sigma_{yy} = 10, \sigma_{xy} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

(i) Fit the equation $y = b_0 + b_1 x_1 + b_2 x_2$ as best linear equation.

(ii) Find the multiple correlation coefficient

(iii) Find the mean square error.

5. (a) If A is a real symmetric matrix of order n, prove that there exists a real orthogonal matrix P of order n such that $A = PDP'$ where D is a real diagonal matrix. Further show that elements of D are eigen values of A and the columns of P are orthonormal eigen vectors of A. 7

(b) In answering a question on a multiple choice test an examinee either knows the answer (with probability p) or guesses (with probability (1-p)). Assume that the probability of answering a question correctly is unity if examinee knows the answer and 1/m if the examinee guesses. Suppose an examinee answers a question correctly, What is the probability that the examinee really know the answer ? 5

(c) Let $X \sim N_3(\mu, \Sigma)$, where $\mu_x = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$ and 3

$\Sigma = \begin{bmatrix} 9 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 6 \end{bmatrix}$. Obtain the conditional

distribution of $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ given $x_3 = 1$.

6. (a) $X = (X_1, X_2, X_3)'$ has $N_P(\mu, \Sigma)$ distribution, 8

where $\mu = (1, -1, 0)'$ and $\Sigma = \begin{pmatrix} 4 & 1 & 2 \\ 1 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$

Find

(i) conditional distribution of X_1 given $X_2 = -0.5$ and $X_3 = 0.2$.

(ii) Obtain a non singular transformation $Y = TX + C$ such that Y_1, Y_2 and Y_3 are independent standard normal variables.

(b) Find principle components and proportion of total population variance explained by each when the covariance matrix is : 7

$$\Sigma = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

7. (a) Consider three random variables X_1, X_2 and X_3 having positive definite covariance matrix given by : 7

$$\Sigma = \begin{bmatrix} 1 & 0.9 & 0.7 \\ 0.9 & 1 & 0.4 \\ 0.7 & 0.4 & 1 \end{bmatrix}.$$

Write its factor model with single factor and show that no proper solution exists in this case.

(b) Determine the definiteness of the following quadratic forms (give details) : 4

(i) $x_1^2 - x_2^2 + x_3^2 - x_1 \ x_2 + 10 \ x_1 \ x_3 - 2x_2 \ x_3$

(ii) $9 \ x_1^2 + 4x_2^2 + 4x_3^2 + x_4^2 + 8 \ x_2 \ x_3 + 12 \ x_3 \ x_1 + 12 \ x_1 \ x_2$

(c) Let the data matrix for a random sample of size $n = 3$ from a bivariate normal population be $X = \begin{bmatrix} 6 & 10 & 8 \\ 9 & 6 & 3 \end{bmatrix}$. 4

Evaluate the observed T^2 for $\mu_0' = [9, 5]$. What is the sampling distribution of T^2 in this case ?

8. State whether the following statements are true or false. Justify your answers : **2x5=10**

(a) Irreducible recurrent Markov Chain always has stationary distributions.

(b) The processes $\{Nt \quad t \geq 0\}$ and $\left\{N_{t+x_1} - 1, t \geq 0\right\}$ have the same distribution.

(c) In the queueing system represented by $M/E_5/3/15/20/FCFS$ the distribution of service time is exponential.

(d) The unique square root of a matrix

$$A = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} \text{ is } \begin{bmatrix} \frac{3}{2} & \frac{\sqrt{7}}{2} \\ -\frac{1}{2} & \frac{\sqrt{7}}{2} \end{bmatrix}.$$

(e) Let $X_i \sim N_p(\mu, \Sigma)$, where $i = 1, 2, 3 \_ \_ \_ N$. be independent, then $\overline{X} \sim N_p(\mu, \Sigma)$.

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MMT-008 (P)

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE) M.Sc. (MACS)

Term-End Practical Examination

June, 2013

MMT-008 (P) : PROBABILITY AND STATISTICS PRACTICAL

Time : 1½ hours *Maximum Marks : 40*

Note : *There are* ***two*** *questions in this paper worth* ***30*** *marks. Remaining* ***10*** *marks are for the* ***viva-voce.***

1. Write a program in C-language to fit the model $y_i = b_0 + b_1 x_{1i} + b_2 x_{2i}$; $1 \leq i \leq n$. You may assume that $n \leq 20$. Use the programme to fit a linear model for the data given below : **20**

y	12	22	30	38	40	25	15	10
x_1	8	3	5	5	17	20	9	5
x_2	1	2	2	5	5	6	6	7

2. Write a program in 'C' language that checks whether a quadratic form in three variables is positive definite or not. It should do the following : **10**

(a) Read the coefficient of the quadratic form.

(b) Print the matrix corresponding to the quadratic form.

(c) Check whether the quadratic form is positive definite or not and print the result.

No. of Printed Pages : 7

MMT-008

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE) M.Sc. (MACS)

Term-End Examination

June, 2013

MMT-008 : PROBABILITY AND STATISTICS

Time : 3 hours *Maximum Marks : 100*

(Weightage : 50%)

Note : *Question number 8 is **compulsory**. Answer **any six** questions from question number 1 to 7. Use of calculator is **not** allowed.*

1. (a) Consider two random variables X and Y whose joint probability mass function is given in the following table. **10**

$x \backslash y$	-5	1	4
1	0.15	0.15	0.2
2	0.05	0.05	0.1
4	0.1	0.1	0.1

(i) Find E(X), E(Y), V(X) and V(Y)

(ii) Are X and Y independent ? Give reasons.

(iii) Find $E(X/Y=4)$ and $V(X/Y=4)$

(iv) Find Cov (X, Y)

(b) Let $X = (X_1 \; X_2 \; X_3)' \sim N_3(\mu, \Sigma)$ where 5

$\mu = (-1 \; 1 \; -1)'$ and $\Sigma = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 4 \end{pmatrix}$. Let

$Y = \begin{pmatrix} 2X_1 + X_2 + X_3 \\ X_1 + 2X_2 + X_3 \end{pmatrix}$. Find $l_{2 \times 1}$ such that

$l'y \sim N(0, 1)$.

2. (a) Consider the mean vector $\mu_X = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and 8

$\mu_Y = 3$ and the covariance matrices of

$(X_1 \; X_2)'$ and Y are $\Sigma_{xx} = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$, $\sigma_{yy} = 14$

and $\sigma_{xy} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

(i) Fit the equation $y = b_0 + b_1 x_1 + b_2 x_2$ as the best linear equation.

(ii) Find the multiple correlation coefficient.

(iii) Find the mean squared error.

(b) Consider a Markov chain with state space $S = \{0, 1, 2, 3\}$ and transition probability matrix $\begin{pmatrix} 1/4 & 0 & 3/4 & 0 \\ 0 & 3/4 & 1/4 & 0 \\ 3/4 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. **7**

(i) Find the communicating classes.

(ii) Find all stationary distributions.

3. (a) Three friends Ashish, Basant and Chetan occupy rooms numbered 1, 2, 3 in a hostel. Another friend Falguni stays with them alternating among the three rooms. He never stays in the same room on two consecutive days. Everyday he chooses one of the two available rooms at random. Let X_n be the room number he stays in on day n. **10**

(i) Show that X_n is a Markov chain.

(ii) Find the transition probability matrix.

(iii) If Falguni starts (on day 0) in room number 3, find the probability that the next time he stays in the same room is on day n.

(iv) Find the mean recurrence time for room 3.

(b) Suppose the interoccurance times $\{x_n : n \geq 1\}$ are uniformly distributed on $[0, 1]$. 5

(i) Find $\widetilde{M}_t$, the Laplace transform of the renewal function, M_t.

(ii) Find $\lim_{t \to \infty} M_{t/t}$.

4. (a) Consider a branching process with offspring distribution given by 7

$$p_j = \begin{cases} 1/3 & j = 0 \\ 2/3 & j = 2 \end{cases}.$$

Find the probability of extinction.

(b) Determine the definiteness of the quadratic form $2x_1^2 + 2x_2^2 + 3x_3^2 + 4x_1 x_3$. 5

(c) Define conjoint analysis. Give two applications of conjoint analysis. 3

5. (a) A particular component in a machine is replaced instantaneously on failure. The successive component lifetimes are uniformly distributed over the interval [2, 5] years. Further, planned replacements take place every 3 years. Compute 6

(i) long - term rate of replacements.

(ii) long - term rate of failures.

(iii) long - term rate of planned replacements.

(b) Suppose $n_1 = 20$ and $n_2 = 30$ observations are made on two variables X_1 and X_2 where $X_1 \sim N_2(\mu^{(1)}, \Sigma)$ and $X_2 \sim N_2(\mu^{(2)}, \Sigma)$ $\mu^{(1)} = (1\ 2)'$, $\mu^{(2)} = (-1\ 0)'$ 9

and $\Sigma = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}'$.

Considering equal cost and equal prior probabilities, classify the observation $(-1\ 1)'$ in one of the two populations.

6. (a) Two samples of sizes 40 and 60 respectively were drawn from two different lots of a certain manufactured component. Two characteristics X_1 and X_2 were measured for the sampled items. The summary statistics of the measurements for lots 1 and 2 is given below. 8

$\overline{X}_1 = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$; $\overline{X}_2 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$; $S_1 = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$;

$S_2 = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$. Assume normality of X_1 and X_2 and that $\Sigma_1 = \Sigma_2$. Test at 1% level of significance whether $\mu_1 = \mu_2$ or not. You may like to use the following values.

$\begin{pmatrix} F_{2,97}(0.01) \approx 4.86 \\ F_{3,97}(0.01) \approx 4.05 \end{pmatrix}$.

(b) Consider three random variables X_1, X_2, X_3 having the following covariance matrix. 7

$$\begin{pmatrix} 1 & 0.12 & 0.08 \\ 0.12 & 1 & 0.06 \\ 0.08 & 0.06 & 1 \end{pmatrix}$$ where no. of variables

$\lambda(p) = 3$ and no. of factors '$(m) = 1$. Write the factor model.

7. (a) Customers arrive at a fast food counter in a Poisson manner at an average of 40 customers per hour. The service time per customer is exponential with mean 1 minute. What is the 10

(i) probability that an arriving customer can go directly to the counter to place the order ?

(ii) probability that there are at least 3 customers in the queue ?

(iii) average queue length ?

(iv) expected waiting time for a customer in the system ?

(v) probability that a customer has to wait at least three minutes in the queue ?

(b) Let N_t be a Poisson process with parameter $\lambda > 0$. Fix $s > 0$ and let the renewal function $M_t = N_{t+s} - N_s$. Show that the conditional distribution of M_t given $N_s = 10$ is Poisson and identify its parameter. 5

8. State whether the following statements are **true** or **false**. Justify your answer with valid reasons : 10

(a) For independent events B and C with $P(B \cap C) > 0$, we have $P(A/B \cap C) = P(A/B).P(A/C)$.

(b) If X and Y are two random variables with $V(X) = V(Y) = 2$, then $-2 < \text{cov}(X, Y) < 2$.

(c) The row sums in the infinitesimal generator of a birth and death process are zero.

(d) A real symmetric matrix $(aij)_{n \times n}$ with $a_{11} = -1$ cannot be positive definite.

(e) The maximum likelihood estimator of $\mu' \Sigma^{-1} \mu$ is $\overline{X}' U^{-1} \overline{X}$ where $\overline{X}$ and U are the maximum likelihood estimators of μ and Σ respectively.

No. of Printed Pages : 2

MMT-008 (P)

00673

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

M.Sc. (MACS)

Term-End Practical Examination

December, 2013

MMT-008 (P) : PROBABILITY AND STATISTICS PRACTICAL

Time : 1½ hours *Maximum Marks : 40*

Note : *There are **two** questions in this paper worth **30** marks. Remaining **10** marks are for the **viva-voce**.*

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1. Write a program in 'C' language to compute the Hotelling's T^2, for any $n \leq 20$. Extend the programme to compute Hotelling's T^2 for the given Data : **20**

$H_0 : \mu' = [7, 11]$ and the data matrix

$$X = \begin{bmatrix} 2 & 12 \\ 8 & 9 \\ 6 & 9 \\ 8 & 10 \end{bmatrix}$$

2. Let $\mathbf{Y} \sim Np\ (\mu, \Sigma)$ **10**

Write a C program to find the mean of the normal variable $\mathbf{Z} = \mathbf{AY}$. When

$$\mathbf{A} = \begin{bmatrix} a_1 \dots a_p \\ b_1 \dots b_p \end{bmatrix}$$

Use the program to find the mean of $\mathbf{Z}$ when

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 2 \end{bmatrix} \text{ and } \mu = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

No. of Printed Pages : 6 **MMT-008**

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE) M.Sc. (MACS)

Term-End Examination 00780

December, 2013

MMT-008 : PROBABILITY AND STATISTICS

Time : 3 hours *Maximum Marks : 100*

(Weightage : 50%)

Note : *Question number 8 is* ***compulsory****. Answer* ***any six*** *questions from question number* **1** *to* **7**. *Use of calculator is* ***not*** *allowed.*

1. (a) Due to a faulty pipe, a certain road gets waterlogged with probability 0.1 on a normal day and with probability 0.7 on a rainy day. It rains approximately half the number of days in July. Ms. Rajni generally reaches office on time 90% of the days if the road is not water logged. But in case of water logging she reaches office on time only 50% of the days. If on a particular day in July, Ms. Rajni was late to office, what is the probability that the road was waterlogged on that day ? **6**

(b) Obtain a spectral decomposition of the matrix $A = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}$ **6**

(c) Consider the mean vector $\mu_x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mu_y = 2$, and the covariance matrices of x_1, x_2 and y are 3

$$\Sigma_{xx} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \ \sigma_{yy} = 9, \ \sigma_{xy} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Fit the equation $y = b_0 + b_1 x_1 + b_2 x_2$ as best linear equation.

2. (a) Consider a Markov chain $\{X_1, X_2 ....\}$ with state space S = {1,2,3}, initial distribution $\pi = (0.1 \ 0.4 \ \ 0.5)$ and transition probability matrix P given by 7

$$P = \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.1 & 0.7 & 0.2 \\ 0.2 & 0.3 & 0.5 \end{pmatrix}$$

(i) Find the probability distribution of X_2.

(ii) Find prob.$(X_1 = X_2 = X_3 = X_4 = 2)$.

(iii) Find prob. $(X_3 = X_4 = 3)$.

(b) Let $X \sim N_3\left(\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{pmatrix}\right)$. 8

Let $Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ where $Y_1 = 2X_1 + X_3$ and $Y_2 = X_2 + 2X_3$.

(i) Find the distribution of Y.

(ii) Find the conditional distribution of Y_1 given $Y_2 = 5$.

3. (a) Consider a branching process with offspring distribution $\{p_j\}$ given by **7**

$$P_j = \begin{cases} \frac{1}{3} & j = 0, 1, 3. \\ 0 & \text{otherwise.} \end{cases}$$

Find the probability of extinction.

(b) Let the data matrix from a random sample of size n=3 from a bivariate normal population be **8**

$$X = \begin{bmatrix} 3 & 7 \\ 8 & 9 \\ 4 & 8 \end{bmatrix}$$

(i) Evaluate T^2 for testing $H_0 : \mu = [7,7]$.

(ii) Test H_0 against $H_1 : \mu \neq [7\ 7]$ at level of significance $\alpha = 0.10$.

(You may use the following tabulated values of F. $F_{2,1}(0.1) = 49.50$, $F_{1,2}(0.1) = 39.86$, $F_{2,3}(0.1) = 5.46$, $F_{3,2}(0.1) = 9.16$.)

4. (a) Aman goes to college by bus. He can use either bus route A or route B to do so. Buses plying on route A arrive in a Poisson fashion with mean λ per hour ($\lambda > 0$), while for route B buses, the time between two consecutive arrivals is exponential with mean $\frac{1}{2\lambda}$ hours. **6**

If buses on routes A and B run independently of each other, and if Aman boards the first bus (A or B) that arrives, what is the probability that Aman has to wait between 10 and 20 minutes at the bus stop ?

(b) Consider two populations π_1 and π_2 having density functions $P_1(x)$ and $P_2(x)$ respectively. Suppose a measurement x_0 is recorded on a new item yielding the density values $P_1(x_0) = 0.2$, $P_2(x_0) = 0.3$. Assign this item to population π_1 or π_2 given the following information. 9

The cost of misclassifying items as π_2 is 75 and misclassifying items as π_1 is 50. Further 30% of all items belong to π_1.

5. (a) Suppose that the lifetimes $X_1, X_2, \ldots$ of a component are i.i.d. with a uniform distribution on [0,10]. Let $0<T<10$ and suppose age replacement policy is to be employed. 6

(i) Find μ^T, the mean renewal time.

(ii) Suppose that each replacement costs 2 units of money. An additional cost of 8 units is incurred if failure occurs. Out of $T=5$ and $T=6$, which one will result in lesser long run average cost per unit time ?

(b) Consider a grocery shop with two cashiers. The average time to prepare a bill and complete payment for a customer is 3 minutes for each cashier. Customers arrive at the cash counters at a rate of 30 per hour, and go to the first available cash counter on a first come first served basis. 9

(i) Find P_0.

(ii) Find the average number of customer in the waiting line.

(iii) Find the average time a customer spends in the queue.

(iv) Find the average time a customer spends in the system.

(v) Find the average number of customers in the system.

6. (a) Find the principal components and proportions of total population variance explained by each component when the covariance matrix is given by 8

$$\Sigma=\begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}$$

(b) Let $\{X_n : n\geq 1\}$ be an i.i.d. Sequence of inter-occurrence times with common p.d.f. $f(x)$ 7

given by $f(x)= \begin{cases} e^{-(x-2)} & \text{if } x > 2 \\ 0 & \text{otherwise} \end{cases}$

(i) Find the Laplace transform $\widetilde{F_t}$ of the distribution function F.

(ii) Find Laplace transform $\widetilde{M_t}$ of the renewal function M_t for the corresponding renewal process.

7. (a) Find the stationary distribution of the Markov chain with transition probability matrix 5

$$P=\begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}.$$

(b) Find the sample correlation matrix for the data matrix 5

$$X=\begin{bmatrix} 4 & 2 \\ 1 & 5 \\ 4 & 8 \end{bmatrix}.$$ Here 3 is the sample size.

(c) Suppose random variables X and Y have joint probability density function f given by **5**

$$f(x,y) = \begin{cases} 8xy & 0<x<y<1 \\ 0 & \text{otherwise} \end{cases}$$

Check whether X and Y are independent.

8. State whether the following statements are true or false. Justify your answer with valid reasons. **10**

(a) For events A1,A2... and B with P(B)>0,

$$P\left(\bigcup_{i=1}^{\infty} Ai \mid B\right) = \sum_{i=1}^{\infty} P(Ai \mid B)$$

(b) If $A = (a_{ij})$ be the matrix of the quadratic form $ax_1^2 + bx_2^2 + cx_1x_2$, then $a_{12} = 2c$.

(c) If N_t be a renewal process for which the inter occurance times X_i's have finite mean, then

$$\lim_{t\to\infty} \frac{Mt}{t} > 0$$

(d) If $X_1, X_2 \ldots X_n$ be independent observations from a population with mean μ and covariance matrix then Σ_{pxp}, then

$n(\overline{X}-\mu)' S^{-1}(\overline{X}-\mu)$ is approximately χ_p.

(e) In a Markov chain if $f_{ii} = 1$ for some state i and i communicates with j, then $f_{jj} = 1$.

No. of Printed Pages : 1

MMT-008 (P)

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

M.Sc. (MACS)

Term-End Practical Examination

00311

June, 2014

MMT-008 (P) : PROBABILITY AND STATISTICS

Time : $1\frac{1}{2}$ hours *Maximum Marks : 40*

Note : *There are **two** questions in this paper, totalling 30 marks. Remaining 10 marks are for the viva-voce.*

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1. Let $y \sim N_3(\boldsymbol{\mu}, \Sigma)$. Write a program in 'C' language to find the distribution of z, which is given as $z = a_1y_1 + a_2y_2 + a_3y_3$. Also, test your program to find the distribution of $z = 4y_1 - 6y_2 + y_3$ for given $\boldsymbol{\mu} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ and $\Sigma = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$. *15*

2. Let the joint probability mass function of x and y be

$$p(x = i, y = j) = \frac{{}^2C_i \; {}^4C_j \; {}^6C_{3-i-j}}{{}^{12}C_3}, \quad 0 \leq i + j \leq 3, \; i = 0, 1, 2, \; j = 0, 1, 2, 3.$$

Write a program in 'C' language to find the marginal distribution of x and y. *15*

No. of Printed Pages : 7

MMT-008

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

M.Sc. (MACS)

00536

Term-End Examination

June, 2014

MMT-008 : PROBABILITY AND STATISTICS

Time : 3 hours *Maximum Marks : 100*
(Weightage : 50%)

Note : *Question no. 8 is* ***compulsory****. Answer any* ***six*** *questions from question numbers 1 to 7. Use of calculator is* ***not*** *allowed.*

1. (a) Joint p.m.f. of random vector (X, Y) is given in the following table :

X \ Y	0	1	2
0	1/8	0	2/8
1	0	1/8	1/8
2	1/8	2/8	0

Find :

(i) marginal p.m.f. of X,

(ii) $P[1 \leq X \leq 2]$ and

(iii) conditional p.m.f. of Y given X = 1. *6*

(b) A post office has two counters. The first handles money-orders, registry and speed post and the second handles all other business. The service times of both the counters follow exponential process with mean 2 minutes. Customers arrive at the first counter 20 per hour and at the second counter 25 per hour in Poisson process. Find the time spent in the post office if both the counters start to handle all the customers. 9

2. (a) Let $\{X_n, n \geq 1\}$ be iid renewal periods with common p.m.f. $P(X_n = 0) = 0{\cdot}4$, $P(X_n = 1) = 0{\cdot}4$ and $P(X_n = 2) = 0{\cdot}2$. Obtain Laplace transform of renewal function $M(t)$. 4

(b) Explain birth and death process, infinitesimal transition rates and generator with the help of one example of each. In a birth and death process $\lambda_k = \lambda$, $\mu_k = \mu$, for all $k \geq 0$, $\lambda, \mu > 0$. Under what condition does stationary distribution exist ? Find the stationary distribution also. 9

(c) Write two advantages and two disadvantages of conjoint analysis. 2

3\. (a) For the Markov chain having following transition matrix :

$$\begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 3/4 & 0 & 1/4 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

find

(i) whether the chain is irreducible ? Give reasons.

(ii) probabilities of ultimate return to the states.

(iii) mean recurrence times to the states. 7

(b) Customers arrive at a ticket counter according to the Poisson process on average 2 every 5 minutes. Service time at the counter follows exponential distribution with mean 2 minutes.

Find :

(i) proportion of time a customer finds empty counter.

(ii) average number of customers at the counter.

(iii) probability that a new customer will have to wait more than 2 minutes at the counter. 8

4\. (a) Explain branching process. In a branching process $\{X_n, n \geq 0\}$ starting from single individual, each individual generates offsprings according to Poisson law. Find p.g.f. of X_n. 7

(b) Explain renewal process and show that a Poisson process is a renewal process also. If the renewal periods in a renewal process are iid exponential then find the distribution of renewal sequence. 8

5. (a) For n = 15 pair of observations from a bivariate normal population, the following summary statistics are obtained :

$$\overline{X} = \begin{bmatrix} 4\cdot5 \\ 8\cdot0 \end{bmatrix} \quad S = \begin{bmatrix} 0\cdot5 & 0\cdot3 \\ 0\cdot3 & 1\cdot7 \end{bmatrix} \quad S^{-1} = \begin{bmatrix} 2\cdot237 & -0\cdot395 \\ -0\cdot395 & 0\cdot658 \end{bmatrix}$$

Test the hypothesis $\bar{\mu}' = [4, 9]$ at 5% level of significance.

[You may like to use the following values $F_{2,\,13}(0\cdot05) = 3\cdot806$), $F_{2,\,14}(0\cdot05) = 3\cdot74$] 8

(b) Let $\mathbf{y} \sim N_3(\bar{\mu}, \Sigma)$ where

$$\mu = \begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix}, \ \boldsymbol{\Sigma} = \begin{pmatrix} 2 & 2 & -1 \\ 2 & 5 & 1 \\ -1 & 1 & 2 \end{pmatrix}$$

Obtain :

(i) distribution of $\mathbf{Cy}$ where

$$\mathbf{C} = \begin{pmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

(ii) linear combination $z = \boldsymbol{l}'\mathbf{y}$ such that $z \sim N(0, 1)$. 7

6. (a) Obtain spectral decomposition of

$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ and evaluate A^4 using the result. 7

(b) Let $\mathbf{y} \sim N_4(\bar{\mu}, \Sigma)$ where $\mu' = (-3 \;\; 2 \;\; -1 \;\; 4)$

and $\Sigma = \begin{pmatrix} 3 & -3 & -1 & 2 \\ -3 & 6 & -1 & -2 \\ -1 & -1 & 2 & 0 \\ 2 & -2 & 0 & 4 \end{pmatrix}$

Obtain :

(i) marginal distribution of $\begin{pmatrix} y_1 \\ y_3 \end{pmatrix}$

(ii) conditional distribution of $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ given $\begin{pmatrix} y_3 \\ y_4 \end{pmatrix}$

(iii) $r_{12}, r_{12.34}$. 8

7. (a) Obtain triangular square root of 7

$A = \begin{bmatrix} 9 & 3 & 3 \\ 3 & 5 & 1 \\ 3 & 1 & 6 \end{bmatrix}$

(b) The variance covariance matrix of three variables X_1, X_2 and X_3 is

$$\Sigma = \begin{pmatrix} 104 & 193 & 105 \\ 193 & 413 & 192 \\ 105 & 192 & 107 \end{pmatrix}$$

The eigenvalues and corresponding eigenvectors are given below :

$\lambda_1 = 602\cdot2 \quad \bar{a}_1' = (0\cdot40, 0\cdot82, 0\cdot40)$

$\lambda_2 = 21\cdot4 \quad \bar{a}_2' = (0\cdot52, -0\cdot57, 0\cdot63)$

$\lambda_3 = 0\cdot4 \quad \bar{a}_3' = (-0\cdot75, 0\cdot04, 0\cdot66)$

(i) Obtain principal components.

(ii) Obtain variances of principal components.

(iii) Verify total variances explained by principal components is equal to total variances of original variables.

(iv) Obtain proportion of variation explained by first two components. *8*

8. State whether following statements are *true* or *false*. Justify your answer. *2×5=10*

(i) Two events A and B are non-null and mutually exclusive, then $P(A|B) = 0$.

(ii) A state i in a Markov chain is persistent, then the series $\sum_{n=0}^{\infty} p_{ii}^{(n)}$ is divergent.

(iii) If P(s) be the p.g.f. of a random variable then P(1) will be its mean.

(iv) When n is large then

$n(\overline{X} - \overline{\mu})' S^{-1} (\overline{X} - \overline{\mu})$ follows

approximately χ_p^2.

(v) Canonical correlation is a particular case of multiple correlation.

No. of Printed Pages : 1

MMT-008(P)

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

M.Sc. (MACS)

Term-End Practical Examination

00244

December, 2014

MMT-008(P) : PROBABILITY AND STATISTICS

Time : $1\frac{1}{2}$ hours *Maximum Marks : 40*

Note : *This question paper is worth 30 marks. Remaining 10 marks are for the viva-voce.*

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1. Let $y \sim N_p(\underset{\sim}{\mu}, \Sigma)$ and $\bar{y} = \dfrac{\sum_{i=1}^{p} y_i}{p}$. Write a program in 'C' language to obtain the distribution of $\bar{y}$. 15

2. Write a program in 'C' language to fit the model $Y_i = b_0 + b_1 X_{1i} + b_2 X_{2i} + e_i$, where i = 1, 2, 3, 4, 5, using the least square estimates. Also, test your program to fit the model for the following data : 15

i	1	2	3	4	5
X_{1i}	8	3	15	17	9
X_{2i}	1	2	2	5	6
Y_i	12	6	3	22	10

No. of Printed Pages : 8

MMT-008

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

M.Sc. (MACS)

00732

Term-End Examination

December, 2014

MMT-008 : PROBABILITY AND STATISTICS

Time : 3 hours *Maximum Marks : 100*

(Weightage : 50%)

Note : *Question no.* ***8*** *is* ***compulsory****. Answer any* ***six*** *questions from questions no.* ***1*** *to* ***7****. Use of calculators is* ***not*** *allowed.*

1. (a) Customers in a bank may get service from any one of two counters. Customers arrive in the bank according to Poisson law at the rate of 15 per hour. Service time at each counter follows exponential distribution with mean 4 minutes. As an alternative, bank thinks to install an automatic servicing machine which has single service counter but it can serve twice faster than a counter clerk. Obtain average waiting time a customer will have to spend in the bank under both the systems and suggest which system will be better. 7

(b) Two random variables X and Y have their joint p.d.f., f(x, y) as given below :

$$f(x, y) = \begin{cases} k(x, y) & 0 < x < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find

(i) the value of k.

(ii) marginal p.d.f. of X and Y.

(iii) and test independence of X and Y. *3*

(c) Let X_1, X_2, X_3 have means 2, 3, 5 and variances 1, 1·5, 1 respectively. If the correlation coefficients $r_{12} = 0{\cdot}5$, $r_{13} = 0{\cdot}4$, $r_{23} = -0{\cdot}7$ then write down mean vector and variance-covariance matrix. *5*

2. (a) The transition matrix P of a Markov chain is given below. Obtain P^n and its limiting value for large n.

$$P = \begin{bmatrix} 0{\cdot}2 & 0{\cdot}8 \\ 0{\cdot}5 & 0{\cdot}5 \end{bmatrix}$$ *5*

(b) Let $X \sim N_4(\mu, \Sigma)$

$$\boldsymbol{\mu} = \begin{bmatrix} 3 \\ -1 \\ 5 \\ 2 \end{bmatrix} \text{ and } \boldsymbol{\Sigma} = \begin{bmatrix} 4 & 0 & 3 & 0 \\ 0 & 9 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \text{ where } \mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}.$$

(i) Obtain marginal distribution of $\begin{bmatrix} X_2 \\ X_4 \end{bmatrix}$.

(ii) Test independence of $\begin{bmatrix} X_1 \\ X_3 \end{bmatrix}$ and $\begin{bmatrix} X_2 \\ X_4 \end{bmatrix}$.

(iii) Find coefficient of correlation between X_2 and X_4. 5

(c) A Markov chain has an initial distribution $\mathbf{u}^{(0)} = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$ and the transition matrix

$$\mathbf{P} = \begin{bmatrix} 0 & 0{\cdot}5 & 0{\cdot}5 \\ 0{\cdot}5 & 0 & 0{\cdot}5 \\ 0{\cdot}5 & 0{\cdot}5 & 0 \end{bmatrix}$$

(i) Is the chain irreducible ? Give reasons.

(ii) Obtain its stationary distribution.

(iii) Is the stationary distribution unique ? Give reasons. 5

3. (a) From the past experience the population mean vector and variance-covariance matrix of (X_1, X_2) are as given below :

$$\boldsymbol{\mu} = \begin{bmatrix} 20 \\ 5 \end{bmatrix}, \ \boldsymbol{\Sigma} = \begin{pmatrix} 10 & 5 \\ 5 & 4 \end{pmatrix}.$$

From a sample of size 10, the sample mean vector was found as

$$\overline{\mathbf{X}} = \begin{bmatrix} 22 \\ 6 \end{bmatrix}$$

Test whether the given population mean is a true representation from the sample information at 5% level of significance. (You may use the values $\chi^2_{2,\,0\cdot05} = 5\cdot99$, $\chi^2_{3,\,0\cdot05} = 7\cdot81$) 6

(b) In a Branching process the offspring distribution follows binomial law as

$$p_k = \binom{n}{k} p^k q^{n-k};$$

$$k = 0, 1, 2, \ldots, n, \ 0 < p < 1, \ p + q = 1.$$

What is the probability of ultimate extinction of the process given that

(i) $n = 2, \ p = 0\cdot3$

(ii) $n = 2, \ p = 0\cdot6$? 6

(c) Find the matrix of the following quadratic form :

$$8x_1^2 + 7x_2^2 + 3x_3^2 - 12x_1x_2 + 4x_1x_3 - 8x_2x_3.$$

And hence identity definiteness of the quadratic form. *3*

4. (a) Suppose $n_1 = 10$ and $n_2 = 15$ observations are taken on variables X_1 and X_2 from two bivariate populations π_1 and π_2. The mean vectors and variance-covariance matrix are given by

$$\mu^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \ \mu^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \ \Sigma^{-1} = \begin{bmatrix} \frac{3}{20} & \frac{1}{20} \\ \frac{1}{20} & \frac{7}{20} \end{bmatrix}.$$

A unit has observation [0, 1]. Classify it to π_1 or π_2 when

(i) costs are equal and prior probabilities are equal.

(ii) prior probabilities are 0·6 and 0·4 and C(1/2) = C(2/1).

(Given is log (2/3) = – 0·4055). *8*

(b) There are three counters at the border of a country to check the passports and other documents. If the arrivals at the border be Poisson with λ per hour and service time of each counter be $\lambda/2$ per hour then (i) what is the probability that all the counters are idle ? (ii) what is the expected length of queue ? *3*

(c) Obtain the renewal equation when the interarrival distribution is uniform on (0, 1). Also, solve the equation for $t < 1$. *4*

5. (a) Let $\{X_n, n = 1, 2, ...\}$ be i.i.d. geometric random variables with probability mass function

$$P(X_n = i) = (1 - p)\, p^{i-1}, \quad i = 1, 2, 3, ...$$

Find the renewal function of the corresponding renewal process. *7*

(b) Suppose the random variables X_1, X_2 and X_3 have the covariance matrix

$$\Sigma = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Find first, second and third principal components. *8*

6. (a) Let the data matrix for a random sample of size $n = 3$ from a bivariate normal population be

$$X = \begin{bmatrix} 6 & 10 & 8 \\ 9 & 6 & 3 \end{bmatrix}.$$

Evaluate the observed T^2 for $\mu_0' = [9, 5]$.

What is the sampling distribution of T^2 ? *9*

(b) Consider the process $\{X(t),\ t \in T\}$ whose probability distribution under a certain condition is given by

$$P\{X(t) = n\} = \frac{(at)^{n-1}}{(1+at)^{n-1}}, \quad n = 1, 2, \ldots$$

$$= \frac{at}{1+at}, n = 0.$$

Find $E(X(t))$ and $var(X(t))$. 6

7. (a) Show that if $\{N_i(t),\ t \geq 0\}$ are independent Poisson processes with rates λ_i, $i = 1, 2$, then $\{N(t),\ t \geq 0\}$ is a Poisson process with rate $\lambda_1 + \lambda_2$, where $N(t) = N_1(t) + N_2(t)$. 8

(b) If $\mathbf{y}$ be $N_3(\mu, \Sigma)$ where

$$\boldsymbol{\mu} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \text{ and } \boldsymbol{\Sigma} = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 9 \end{bmatrix}$$

find

(i) distribution of $\mathbf{z} = 2y_1 - y_2 + y_3$

(ii) distribution of $\mathbf{z} = \begin{bmatrix} y_1 + y_2 - y_3 \\ 2y_1 - y_2 + y_3 \end{bmatrix}$

(iii) $r_{12}, r_{13}, r_{23}, r_{23.1}$ 7

8. State whether the following statements are *true* or *false*. Justify your answers with valid reasons. *10*

(a) The relation of accessibility in states is transitive.

(b) Let X_1 and X_2 be bivariate normal variables and are uncorrelated, then X_1 and X_2 are not necessarily independent.

(c) If $A = \begin{pmatrix} 3 & -1 \\ -1 & -2 \end{pmatrix}$, then A is positive semi-definite.

(d) In a renewal process the durations between successive occurrences of events are not necessarily independent.

(e) In a Markov chain a state will be persistent if in a long run the return to the state is certain.

No. of Printed Pages : 1

MMT-008(P)

00123

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

M.Sc. (MACS)

Term-End Practical Examination

June, 2015

MMT-008(P) : PROBABILITY AND STATISTICS

Time : $1\frac{1}{2}$ hours *Maximum Marks : 40*

Note : (i) *There are **two** questions in this paper worth 30 marks. Both the questions are **compulsory**.*

(ii) *Remaining 10 marks are for the viva-voce.*

(iii) *All the symbols used have their usual meaning.*

1. Write a program in 'C' language to fit the model $y_i = b_0 + b_1 x_{1i} + b_2 x_{2i}$, $1 \le i \le n$. You may assume that $n \le 20$. Use the program to fit a linear model for the data given below : *15*

y	12	22	30	38	40	25	15	10
x_1	5	9	20	17	5	5	3	8
x_2	7	6	6	5	5	2	2	1

2. Consider $N_4(\mu, \Sigma)$, where

$$\mu = \begin{bmatrix} 2 \\ 4 \\ 1 \\ -3 \end{bmatrix} \text{ and } \Sigma = \begin{bmatrix} 9 & 0 & 2 & 0 \\ 0 & 4 & 0 & 1 \\ 2 & 0 & 6 & 0 \\ 0 & 1 & 0 & 9 \end{bmatrix}.$$

Write a program in 'C' language to obtain the conditional distribution of $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ given $\begin{bmatrix} y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1{\cdot}2 \\ -2{\cdot}6 \end{bmatrix}$. *15*

No. of Printed Pages : 7

MMT-008

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

M.Sc. (MACS)

Term-End Examination

00558

June, 2015

MMT-008 : PROBABILITY AND STATISTICS

Time : 3 hours *Maximum Marks : 100*

(Weightage : 50%)

Note : *Question no. 8 is **compulsory**. Answer any **six** questions from questions no. **1** to **7**. Use of calculator is **not** allowed.*

---

1. (a) Consider M/M/1 queueing system with arrival rate λ and service rate 2λ and M/M/2 queueing system with arrival rate λ and service rate λ. Show that the average waiting time in system M/M/1 is smaller than the waiting time in M/M/2 system. *7*

(b) Obtain the renewal equation when the inter arrival distribution is uniform on [0, 1]. Solve the equation for $t < 1$ also. *4*

(c) Verify Chapman-Kolmogorov equation from the following transition matrix of a Markov Chain for m = n = 1 : *4*

$$P = \begin{bmatrix} 0{\cdot}5 & 0{\cdot}5 & 0 \\ 0 & 0{\cdot}5 & 0{\cdot}5 \\ 0{\cdot}5 & 0 & 0{\cdot}5 \end{bmatrix}.$$

2. (a) Determine the closed sets and mean recurrence times of the states of the Markov Chain with the following transition matrix : *7*

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 3/4 & 0 & 1/4 \\ 0 & 1 & 0 \end{bmatrix} \end{array}.$$

(b) Let $\mathbf{X} \sim N_3(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ where

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}, \ \boldsymbol{\mu} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} \text{ and } \boldsymbol{\Sigma} = \begin{pmatrix} 9 & 0 & 3 \\ 0 & 4 & 2 \\ 3 & 2 & 9 \end{pmatrix}.$$

(i) Find the marginal distribution of $\begin{pmatrix} X_1 \\ X_3 \end{pmatrix}$.

(ii) Conditional distribution of X_1, given $X_2 = 1$ and $X_3 = 2$.

(iii) Coefficient of correlation between X_1 and X_3. 8

3. (a) Write the postulates of the birth and death process. Obtain the infinitesimal generator for birth and death process with birth rates λ_k and death rates μ_k. 7

(b) Two populations π_1 and π_2 have identical variances. A sample of size 40 was drawn from π_1 and a sample of size 50 was drawn from π_2. The summary statistics were

$$\bar{x}_1 = \begin{bmatrix} 7 \\ 3 \end{bmatrix}, \qquad \bar{x}_2 = \begin{bmatrix} 9 \\ 2 \end{bmatrix}$$

$$S_1 = \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}, \qquad S_2 = \begin{bmatrix} 2 & 1 \\ 1 & 6 \end{bmatrix}.$$

Test equality of means of both populations at 5% level of significance.

(You may use the values $F(2, 87)_{.05} = 3{\cdot}10$, $F(3, 87)_{.05} = 2{\cdot}71$) 8

4. (a) One of the two teller machines handles withdrawals only while the other handles deposits only in a bank. The service time of both the machines follows exponential distribution with mean service time 3 minutes. The depositors arrive in the bank at the rate of 16 per hour and withdrawers arrive at the rate of 14 per hour in Poisson fashion. Find the average waiting times of depositors and withdrawers in the queue. If each machine can handle both the jobs of deposits and withdrawals, then what will be the average waiting time in queue for a customer ? *9*

(b) Let $X = (X_1\ X_2\ X_3)' \sim N_3(\mu, \Sigma)$, where

$$\mu = (-1\ \ 1\ \ -1)' \text{ and } \sum = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 4 \end{bmatrix}.$$

$$\text{Let } Y = \begin{pmatrix} X_1 + X_2 + X_3 \\ X_1 + 2X_2 \end{pmatrix}.$$

Find l_2, such that $l' Y \sim N(0, 1)$. *6*

5. (a) Consider the mean vectors be $\mu_x = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $\mu_y = 4$, and the covariance matrices of x_1, x_2 and y are

$$\sum\nolimits_{xx} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix},\ \sigma_{yy} = 9,\ \sigma_{xy} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

(i) Fit the equation $y = b_0 + b_1x_1 + b_2x_2$ as best linear fit equation.

(ii) Find the multiple correlation coefficient.

(iii) Find the mean square error. 8

(b) Consider a branching process with offspring distribution given by

$$p_j = \begin{cases} 2/5, & j = 0 \\ 3/5, & j = 2. \end{cases}$$

Find the probability of extinction. 7

6. (a) Consider two random variables X and Y whose joint probability mass function is given in the following table :

X \ Y	4	3	2
1	0·15	0·15	0·2
2	0·05	0·05	0·1
4	0·1	0·1	0·1

(i) Find E(X), E(Y), V(X) and V(Y).

(ii) Are X and Y independent ? Give reasons.

(iii) Find E(X/Y = 4) and V(X/Y = 4).

(iv) Find cov (X, Y). 10

(b) A particular component in a machine is replaced instantaneously on failure. The successive component lifetimes are uniformly distributed over the interval [1, 4] years. Further, planned replacements take place every 3 years. Compute

(i) long-term rate of replacements,

(ii) long-term rate of failures,

(iii) long-term rate of planned replacements. 5

7. (a) The variance-covariance matrix of three random variables X_1, X_2, X_3 is given by

$$\sum = \begin{bmatrix} 1 & 0{\cdot}63 & 0{\cdot}4 \\ 0{\cdot}63 & 1 & 0{\cdot}35 \\ 0{\cdot}4 & 0{\cdot}35 & 1 \end{bmatrix}.$$

Write its factor model. 9

(b) Consider a 3-state Markov Chain with the transition matrix.

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

Find the stationary distribution of it. 6

8. State whether the following statements are *true* or *false*. Justify your answer. 10

(a) The matrix $\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$ is a variance-covariance matrix of two dimensional random variable.

(b) The quadratic form $x_1^2 - x_2^2$ is positive definite.

(c) The time between successive arrivals follows exponential distribution each with mean time 10 minutes, then number of arrivals follows Poisson distribution with mean 10 per hour.

(d) A subset C of the state space of a Markov Chain is closed, then any state in C can communicate with a state outside C.

(e) Principal components depend on the scales used to measure the variables.

No. of Printed Pages : 1

MMT-008(P)

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

M.Sc. (MACS)

00353

Term-End Practical Examination

December, 2015

MMT-008(P) : PROBABILITY AND STATISTICS

Time : $1\frac{1}{2}$ hours *Maximum Marks : 40*

Note : *(i) There are **two** questions in this paper worth 30 marks. Both the questions are **compulsory**.*

(ii) Remaining 10 marks are for the viva-voce.

(iii) All the symbols used have their usual meaning.

1. Consider $Y = [y_1 \quad y_2 \quad y_3]'$ having $N_3(\mu, \Sigma)$, where

$$\mu = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} \text{ and } \Sigma = \begin{bmatrix} 9 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 6 \end{bmatrix}.$$

Write a program in 'C' language to find the marginal distribution of y_1, y_2 and y_3. Also, extend this program to find the conditional distribution of y_1, given y_2 and y_3. *20*

2. Write a program in 'C' language that checks whether a quadratic form in three variables is positive definite or not. It should do the following : *10*

(i) Read the coefficients of the quadratic form.

(ii) Print the matrix corresponding to the quadratic form.

(iii) Check whether the quadratic form is positive definite or not and print the result.

No. of Printed Pages : 8

MMT-008

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

01254

M.Sc. (MACS)

Term-End Examination

December, 2015

MMT-008 : PROBABILITY AND STATISTICS

Time : 3 hours *Maximum Marks : 100*

(Weightage : 50%)

Note : *Question no. 8 is **compulsory**. Answer any **six** questions from questions no. **1** to **7**. Use of calculator is **not** allowed.*

---

1. (a) Consider a closed queuing network with three queues with exponential services. Service rates for these queues are μ_1, μ_2 and μ_3. The total number of clients in the queue is N = 3. Routing probabilities between the queues are defined by the following matrix :

$$[p_{ij}] = \begin{bmatrix} 0 & 0{\cdot}5 & 0{\cdot}5 \\ 0{\cdot}5 & 0{\cdot}5 & 0 \\ 0{\cdot}5 & 0{\cdot}5 & 0 \end{bmatrix}$$

(i) Draw the queuing system.

(ii) Find the steady state probabilities.

(iii) If $\mu_1 = \mu_2 = \mu_3 = \mu$, then find the average number of customers in steady state and also find the average time spent by the customers in each queue. 7

(b) A fair die is tossed and its outcome is denoted by X, i.e.

$$X \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix}$$

After that, X independent fair coins are tossed and the number of heads obtained is denoted by Y.

Find : 8

(i) $P[Y = 4]$

(ii) $P[X = 5 \mid Y = 4]$

(iii) $E(Y)$

(iv) $E(XY)$

2. (a) Find the principal components and proportions of total population variance explained by each component when the covariance matrix is given by

$$\Sigma = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}.$$ 6

(b) Let $X \sim N_4(\mu, \Sigma)$ with

$$\mu = \begin{bmatrix} 2 \\ 4 \\ -1 \\ 3 \end{bmatrix} \text{ and } \Sigma = \begin{bmatrix} 4 & 2 & -3 & 4 \\ 2 & 4 & 0 & 1 \\ -3 & 0 & 4 & -2 \\ 4 & 1 & -2 & 8 \end{bmatrix}.$$

Let Y and Z be two partitioned subvectors of X such that $Y' = [X_1, X_2]$ and $Z' = [X_3, X_4]$.

Find : 9

(i) $E(Y/Z)$

(ii) $Cov(Y, Z)$

(iii) $r_{12.34}$

3. (a) The transition probability matrix P of a Markov chain with states Sunny (S), Cloudy (C) and Rainy (R) in a simple weather model is given below :

$$P = \begin{array}{c} \\ S \\ C \\ R \end{array} \begin{array}{c} \begin{array}{ccc} S & C & R \end{array} \\ \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.5 & 0.3 \\ 0.1 & 0.5 & 0.4 \end{bmatrix} \end{array}$$

Also the initial probability distribution of the states is (0·5 0·3 0·2).

(i) What is the probability that starting from initial day, all the three successive days will be cloudy ?

(ii) Obtain the probability distribution of the weather on the second day. 5

(b) A radioactive source emits particles at a rate of 5 per minute according to Poisson law. Each particle emitted has probability 0·6 of being recorded. What is the probability that in 4 minutes 10 particles will be recorded ? What is the mean and variance of the number of particles recorded ? 5

(c) Let the lifetimes X_1, X_2, ... be i.i.d. exponential random variables with parameters $\lambda > 0$ and $T > 0$. Age replacement policy is to be employed.

(i) Find μ^T.

(ii) If the replacement cost $C_1 = 3$ and extra cost $C_2 = 4$, then find the long run average cost per unit time. 5

4. (a) Let $\{X_n, n = 1, 2, ...\}$ be i.i.d. geometric random variables with the probability mass function

$$P(X_n = i) = (1 - p)p^{i-1}, i = 1, 2, 3, ...$$

Find the renewal function of the corresponding renewal process. 7

(b) Let $\bar{X} = \{X_1, X_2, X_3\}$ be a random vector and X be the data matrix given below :

$$X' = \begin{bmatrix} 5 & 2 & 5 \\ 3 & 4 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

Find : 8

(i) Variance – Covariance matrix Σ

(ii) Correlation matrix R

5. (a) On the basis of past experience about the sales (X_1) and profits (X_2) the population mean vector and variance – covariance matrix for the industry was as given below :

$$\mu = \begin{bmatrix} 30 \\ 10 \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} 10 & 5 \\ 5 & 4 \end{bmatrix}$$

From a sample of 10 industries the sample mean vector was found as $\bar{X} = \begin{bmatrix} 33 \\ 7 \end{bmatrix}$. Test at 5% level of significance whether the sample confirms the truthfulness of the industry claim of population mean vector. [You may like to use the following values : $x^2_{2,\, 0\cdot05} = 5\cdot99$, $x^2_{3,\, 0\cdot05} = 7\cdot81$] 7

(b) Suppose $X = (X_1, X_2, X_3)'$ be distributed as a trivariate normal distribution, $N_3(\mu, \Sigma)$, where

$$\mu = (2, 1, 3)', \quad \Sigma = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}.$$

Find the distribution of

$$u = \begin{bmatrix} X_1 - X_3 \\ X_1 + X_3 - 2X_2 \end{bmatrix}.$$ 8

6. (a) In a Branching process, the offspring distribution is given as

$$P_k = {}^nC_k \, p^k \, q^{n-k}, \; k = 0, 1, 2, \ldots, n$$
$$q = 1 - p,$$
$$0 < p < 1.$$

Find the probability of ultimate extinction of the process given that

(i) $n = 2, p = 0{\cdot}2,$

(ii) $n = 2, p = 0{\cdot}8.$ 7

(b) If $\{ X(t) \; ; \; t > 0 \}$ is a Poisson process with rate λ and S_m denotes the duration from start to the occurrence of m^{th} event, obtain the distribution of S_m. If $\lambda = 1$ per hour, then find the probability that the duration from start to the occurrence of third event will be less than 2 hours. 8

7\. (a) Suppose that the random variables X and Y have the following joint p.d.f. :

$$f(x, y) = \begin{cases} x + y ; & 0 < x < 1,\ 0 < y < 1 \\ 0 \ ; & \text{otherwise} \end{cases}$$

(i) Find the conditional p.d.f. of X given $Y = y$.

(ii) Check independence of X and Y.

(iii) Compute $P\left[0 < X < \frac{1}{3} \middle| Y = \frac{1}{2}\right]$. 7

(b) Let X denote the data matrix for a random sample of size 3 from a bivariate normal population, where

$$X = \begin{bmatrix} 6 & 9 \\ 10 & 6 \\ 8 & 3 \end{bmatrix}.$$

Test the hypothesis $H_0 : \mu = (9, 5)'$ at 5% level of significance. [You may like to use the following values : $F_{2,\ 1,\ 0\cdot05} = 18\cdot51$, $F_{3,\ 2,\ 0\cdot05} = 9\cdot55$] 8

8. State whether the following statements are *true* or *false*. Give a short proof or counter-example in support of your answer. 10

(a) If $f_{ij} < 1$ and $f_{ji} < 1$, then i and j do not intercommunicate.

(b) If P(s) is the generating function of the random variable X, then the generating function of 2X + 1 is 2P(s) + 1.

(c) If $\{n(t), t > 0\}$ is a Poisson process with rate λ, then $E[n(t+s) - n(s)] = \lambda t$.

(d) The quadratic form $Q = 2x_1^2 - 3x_2^2 - 6x_1x_2$ is negative definite.

(e) If $X \sim N_p(\mu, \Sigma)$, then the linear combinations of the components of X are normally distributed.

No. of Printed Pages : 2

MMT-008 (P)

00280

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE) M.Sc. (MACS)

Term-End Practical Examination

June, 2016

MMT-008 (P) : PROBABILITY AND STATISTICS

Time : 1½ hours *Maximum Marks : 40*

Note : *There are two questions in this paper worth* ***30*** *marks. Remaining* ***10*** *marks are for the viva-voce.*

1. Let $X \sim N_p(\mu, \Sigma)$. Write a programme in 'C' language to obtain the distribution of $Y = CX$, where **20**

$$C = \begin{bmatrix} a_1 & a_2 & \dots a_p \\ b_1 & b_2 & \dots b_p \end{bmatrix}$$

Use the programme to find the distribution of Y,

when $C = \begin{bmatrix} 2 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$, $\mu = \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix}$ and $\Sigma = \begin{bmatrix} 6 & 1 & 2 \\ 1 & 8 & 4 \\ 2 & 4 & 9 \end{bmatrix}$.

2. Consider the mean vectors $\mu_x = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $\mu_y = 4$, and the covariance matrices of x_1, x_2 and y are $\Sigma_{xx} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, $\sigma_{yy} = 9$ and $\sigma_{xy} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. **10**

Write a programme in 'C' language to fit the equation $y = b_0 + b_1 x_1 + b_2 x_2$ as best linear equation.

No. of Printed Pages : 8

MMT-008

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

M.Sc. (MACS)

00346

Term-End Examination

June, 2016

MMT-008 : PROBABILITY AND STATISTICS

Time : 3 hours *Maximum Marks : 100*

(Weightage : 50%)

Note : *Question no. 8 is* ***compulsory****. Answer any* ***six*** *questions from questions no. 1 to 7. Use of calculator is* ***not*** *allowed.*

1. (a) Consider a closed queuing system with 3 stations and 2 clients. Stations are in cascade and from the last queue, clients recycle to the first one. All queues have exponential services have no queuing line, thus, when a client arrives to one of the first two stations and finds the server occupied, he proceeds immediately to the following queue.

    (i) Find the steady state distribution, if it exists.

    (ii) Find the average number of clients in each of the queues.

    (iii) Find the average time spent by clients in each of the queues. *7*

(b) Let $\{X_n\}_{n \in N_0}$ be a Markov chain with the transition matrix

$$P = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

(i) Find all stationary distributions.

(ii) If the chain starts from the state $i = 1$, what is the expected number of steps before it returns to 1 ?

(iii) How many times, on an average, does the chain visit 2 between two consecutive visits to 1 ? 8

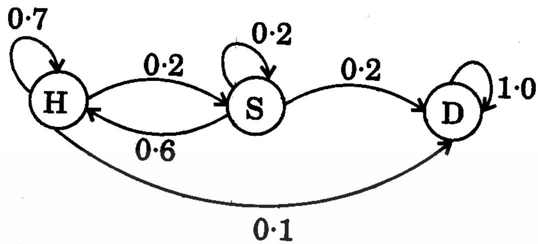
2. (a) Let $\{Z_n\}_{n \in N_0}$ be a Branching process with state space $S = \{0, 1, 2, 3, 4, ...\} = N_0$, and the probability of each offspring is 2. Classify the states and describe all closed sets. 5

(b) Consider the orthogonal transformation of the correlated zero mean random variables x_1 and x_2 and consider

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

If $E(x_1^2) = \sigma_1^2$, $E(x_2^2) = \sigma_2^2$ and $E(x_1 x_2) = \rho\sigma_1\sigma_2$, determine the angle θ such that y_1 and y_2 are uncorrelated. 4

(c) The transition graph of a Markov chain having three states, healthy, sick and dead, which provides transition probabilities for the changes in a week in the condition of a patient, is given below :



If the probability that a patient is healthy be 0·8, then

(i) find the probability that he/she will be sick in the coming first week.

(ii) find the probability that he/she will remain healthy in the next two weeks.

Also write the transition probability matrix. 6

3. (a) State the assumptions of the Poisson process. Taxis arrive at a spot from north at a rate of 40 per hour and from south at a rate of 60 per hour in accordance with independent Poisson process. Find the probability that a person will have to wait for a taxi at the spot more than 2 minutes. How many taxis will arrive at the spot in 10 minutes on an average ? 6

(b) Joint probability distribution of two random variables X_1 and X_2 are given in the following table :

X_1 \ X_2	0	1
–1	0·16	0·14
0	0·04	0·26
1	0·10	0·30

Find

(i) mean vector,

(ii) variance-covariance matrix,

(iii) correlation matrix. 9

4\. (a) A Markov chain has the following transition matrix :

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

Determine the probabilities of ultimate return to the states and mean recurrence times of the states. Check whether the chain is irreducible. 6

(b) A random sample of size 4 from $N_2(\mu, \Sigma)$ is given below :

x_1	2	8	6	8
x_2	12	9	9	10

Test the hypothesis $H_0 : \mu' = [7, 11]$ against $H_1 : \mu' \neq [7, 11]$ at 5% level of significance.

[You may like to use the values

$F_{2, 4}(0{\cdot}05) = 19{\cdot}25$ and $F_{2, 2}(0{\cdot}05) = 19{\cdot}00$] 9

5. (a) Variables x_1, x_2 and x_3 have the following variance-covariance matrix :

$$\Sigma = \begin{bmatrix} 1 & 0{\cdot}63 & 0{\cdot}4 \\ 0{\cdot}63 & 1 & 0{\cdot}35 \\ 0{\cdot}4 & 0{\cdot}35 & 1 \end{bmatrix}$$

Write its factor model. 9

(b) Suppose in a branching process, the offspring distribution is given as

$$p_k = {}^nC_k\, p^k\, q^{n-k};\ 0 < p < 1,\ q = 1 - p,$$
$$k = 0, 1, 2, \ldots$$

What will be the probability of extinction of this branching process ? 3

(c) Suppose that families migrate to an area at a Poisson rate $\lambda = 2$ per week. The number of people in each family is independent and takes the values 1, 2, 3, 4 with respective probabilities $\frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{6}$. Find the expected value and variance of the number of individuals migrating to this area during a fixed five-week period. 3

6. (a) Let $[X_1, X_2, X_3]$ be distributed as $N_3(\mu, \Sigma)$,

where $\mu = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ and $\Sigma = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 2 \end{bmatrix}$.

Find

(i) the conditional densities $f(x_2/x_1, x_3)$ and $f(x_1, x_2/x_3)$,

(ii) the distribution of $z = 2x_1 + x_2 - x_3$. 8

(b) Calculate the least square estimate b, the residual e and the residual sum of squares for a straight-line model $Y = b_0 + b_1X + e$ fitted to the following data : 7

X	0	1	2	3	4
Y	1	4	3	8	9

7. (a) Let X_1, X_2 and X_3 be random variables with means 3·3, – 3·1 and 2·5 respectively and the variances 9, 16 and 25 respectively. Also let $\rho_{12} = 0{\cdot}5$, $\rho_{13} = 0{\cdot}3$, $\rho_{23} = -0{\cdot}4$, where ρ_{ij} is the correlation coefficient between x_i and x_j.

(i) Write down the mean vector and the variance-covariance matrix of $X = [X_1 \;\; X_2 \;\; X_3]^t$.

(ii) Find the mean and variance of $1^t x$,

where $1 = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(iii) Also, find the covariance between $1^t x$

and $m^t x$, where $m = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, 1 = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. *10*

(b) Let $X = [X_1 \;\; X_2 \;\; X_3]$ having the covariance

matrix $\Sigma = \begin{bmatrix} 4 & 4 & -2 \\ 4 & 9 & 1 \\ -2 & 1 & 16 \end{bmatrix}$.

Find the correlation matrix. *5*

8. State whether the following statements are *true* or *false*. Give short proof or counter-example to support your answer. 10

(a) In a Markov chain $\{X_n\}_{n \in N_0}$ with state space S, if all rows of transition matrix are equal, then all states belong to the same class.

(b) If (X, Y, Z) be a trivariate random variable, where X, Y and Z are independent uniform random variables over (0, 1), then $P(Z \geq XY) = \frac{3}{4}$.

(c) If the joint pdf of a bivariate random variable (X, Y) is

$$f_{XY}(x, y) = \frac{1}{2\sqrt{3}\pi} \exp\left[-\frac{1}{2}(x^2 - xy + y^2 + x - 2y + 1)\right]$$

$$-\infty < x, y < \infty,$$

then the var(X, Y) = 1 and coefficient of correlation is $\frac{1}{2}$.

(d) If A is a 3×3 transition probability matrix, then the sum of the entries of the matrix A^3 is 3.

(e) A finite Markov chain has a null persistent state.

No. of Printed Pages : 8

MMT-008

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)
M.Sc. (MACS)

00774

Term-End Examination

December, 2016

MMT-008 : PROBABILITY AND STATISTICS

Time : 3 hours — *Maximum Marks : 100*

(Weightage : 50%)

Note : *Question no. 8 is **compulsory**. Answer any **six** questions from questions no. 1 to 7. Use of calculator is **not** allowed. All the symbols used have their usual meaning.*

1. (a) Arrivals at a counter in a bank occur in accordance with Poisson law having average rate of 10 per hour. The service time of a customer follows exponential law with mean time 5 minutes. Find the following : 5

   (i) Average number of customers at the counter including the one that is being served.

   (ii) Average time required in getting service to a new arrival.

(iii) Probability that a new arrival finds the counter empty when he/she arrives.

(iv) Probability that a new arrival finds at least five customers at the counter.

(b) A bag contains 4 red and 6 black balls. Two red balls and one black ball were marked for superior quality. One ball was chosen from the bag and the ball was marked. What is the probability that the ball was red ? *4*

(c) Consider the random vector $X' = (X_1, X_2, X_3)$ having the following covariance matrix :

$$\Sigma = \begin{bmatrix} 1 & 0{\cdot}5 & 0{\cdot}5 \\ 0{\cdot}5 & 1 & 0{\cdot}25 \\ 0{\cdot}5 & 0{\cdot}25 & 1 \end{bmatrix}$$

Write its factor model with one underlying factor. *6*

2\. (a) The joint probability density function of continuous random variables X and Y is given below :

$$f(x, y) = \begin{cases} x + y, & 0 < x < 1,\ 0 < y < 1 \\ 0, & \text{elsewhere} \end{cases}$$

(i) Find the marginal probability density functions of X and Y.

(ii) Test independence of X and Y.

(iii) Compute $P[X > 0{\cdot}1]$.

(iv) Find the conditional probability density function of X, given $Y = 0{\cdot}3$. 7

(b) A sample of 10 industrial corporations observed for their sales (X_1) and profits (X_2) has provided the following information :

$$\bar{X} = \begin{bmatrix} 33 \\ 7 \end{bmatrix} \quad S^{-1} = \begin{bmatrix} 0{\cdot}01 & -0{\cdot}03 \\ -0{\cdot}03 & 0{\cdot}25 \end{bmatrix}$$

Test whether the average industrial sales and profits may be accepted at 5% level of significance.

Given $\mu = \begin{bmatrix} 40 \\ 10 \end{bmatrix}$. [You may like to use the values $F_{2, 8, 0{\cdot}05} = 4{\cdot}46$, $F_{2, 10, 0{\cdot}05} = 19{\cdot}40$] 6

(c) Consider a renewal process whose mean-value function is given by $m(t) = 2t$, $t \geq 0$. What is the distribution of the number of renewals occurring by time 10 ? 2

3. (a) Let X be $N_3(\mu, \Sigma)$ with

$$\Sigma = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Check the independence of the following random variables and justify your answer : *8*

(i) X_1 and X_2

(ii) (X_1, X_2) and X_3

(iii) $(X_1 + X_2)$ and X_3

(b) Suppose 10 and 15 observations are made on random variables X_1 and X_2, respectively, from two populations π_1 and π_2. Let $\mu^{(1)} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\mu^{(2)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\Sigma^{-1} = \begin{bmatrix} 0{\cdot}15 & 0{\cdot}05 \\ 0{\cdot}05 & 0{\cdot}15 \end{bmatrix}$.

Considering equal costs and equal prior probabilities, check whether the observation [3, 1] belongs to population π_1 or π_2. *7*

4. (a) In a two-state simple weather model, the probability of a dry day following a rainy day is 0·2 and the probability of a rainy day following a dry day is 0·1.

   (i) Write the transition probability matrix P for this Markov chain.

   (ii) Find $P^{(3)}$.

   (iii) If the probability of a dry day on 15^{th} July, is 0·3, then find the probability that 18^{th} July will be a dry day. *8*

   (b) Define a renewal process. If the renewal process $\{N_t, t = 0, 1, 2, ...\}$ is a negative binomial process, then obtain its renewal function. *4*

   (c) Suppose that p(x, y), the joint probability mass function of X and Y, is given by $p(1, 1) = 0{\cdot}5$, $p(1, 2) = 0{\cdot}1$, $p(2, 1) = 0{\cdot}1$, $p(2, 2) = 0{\cdot}3$. Calculate the probability mass function of X, given that Y = 1. *3*

5. (a) For a Poisson process $\{X(t) : t \geq 0\}$, find the probability that there are k events in time t, given that there are k + n events in time t + s. *4*

(b) Let $X = (X_1, X_2, X_3)'$ follow $N_3(\mu, \Sigma)$, where

$$\mu = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \text{ and } \Sigma = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}.$$

(i) Write down the marginal distributions of X_2 and X_3.

(ii) Write down the marginal distribution of $\begin{bmatrix} X_2 \\ X_3 \end{bmatrix}$.

(iii) Obtain the conditional distribution of $\begin{bmatrix} X_2 \\ X_3 \end{bmatrix}$, given $X_1 = 3$. *8*

(c) Write any three applications of Conjoint Analysis. *3*

6. (a) Let $X' = (X_1, X_2, X_3)$ be a random vector and the following be the data matrix of X :

$$X = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & 4 \\ 2 & 2 & 5 \end{bmatrix} \cdot \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

Find

(i) variance-covariance matrix,

(ii) correlation matrix. *9*

(b) In a branching process, the offspring distribution is given as

$$p_k = \binom{n}{k} p^k q^{n-k} ; k = 0, 1, 2, ..., n,$$

$$q = 1 - p, 0 < p < 1.$$

Find the probability of ultimate extinction of the process when

(i) $n = 2, p = 0 \cdot 3$

(ii) $n = 2, p = 0 \cdot 8$ *6*

7. (a) A Markov chain has the following transition matrix :

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(i) Find the stationary distributions of the chain.

(ii) How many distributions are possible ?

(iii) How many closed sets are there in the chain ? *6*

(b) Consider a system of two servers where customers from outside the system arrive at Server 1 at a Poisson rate 4 and at Server 2 at a Poisson rate 5. The service rates of Servers 1 and 2 are 8 and 10, respectively. A customer upon completion of service at Server 1 is equally likely to go to Server 2 or to leave the system, whereas, a departure from Server 2 will go 25 percent of the time to Server 1 and will depart the system otherwise. Determine the limiting probabilities. *9*

8. State whether the following statements are *True* or *False*. Justify your answers. *2×5=10*

(a) If P is a 4×4 transition probability matrix, then the sum of all elements of P is 4.

(b) If at a doctor's chamber the arrival rate is larger than departure rate, then the number of patients in the chamber will not change in the visiting hour.

(c) Bayes' theorem improves the probability of an event using information on the happening of other event.

(d) Principal components of a set of variables do not depend upon the scales used to measure the variables.

(e) In age replacement policy, the component is replaced at fixed time points T, 2T, 3T, ...

No. of Printed Pages : 2

MMT-008(P)

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE) M.Sc. (MACS)

00248

Term-End Practical Examination

December, 2016

MMT-008(P) : PROBABILITY AND STATISTICS

Time : $1\frac{1}{2}$ Hours *Maximum Marks : 40*

Note : *(i) There are two questions in this paper worth 30 marks.*

*(ii) Answer **both** of them.*

(iii) Remaining 10 marks are for viva-voce.

1. Let $\mathbf{X} \sim N_4(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where

$$\boldsymbol{\mu} = \begin{bmatrix} 2 \\ 4 \\ 1 \\ -3 \end{bmatrix} \text{ and } \boldsymbol{\Sigma} = \begin{bmatrix} 9 & 0 & 2 & 0 \\ 0 & 4 & 0 & 1 \\ 2 & 0 & 6 & 0 \\ 0 & 1 & 0 & 9 \end{bmatrix}.$$

Write a program in 'C' language to find

(a) Marginal distribution of $\begin{bmatrix} x_1 \\ x_3 \end{bmatrix}$

(b) Conditional distribution of $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ given $\begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1{\cdot}2 \\ -2{\cdot}6 \end{bmatrix}$

(c) Correlation coefficient between x_1 and x_3. *15*

2. Write a program in 'C' language to compute the Hotelling's T^2, for any $n \leq 10$. Extend the program to compute Hotelling's T^2 for the given data : *15*

$$H_0 : \boldsymbol{\mu}' = [7, 11] \text{ and the data matrix } \mathbf{X} = \begin{bmatrix} 2 & 12 \\ 8 & 9 \\ 6 & 9 \\ 8 & 10 \end{bmatrix}.$$

No. of Printed Pages : 1

MMT-008(P)

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)
M.Sc. (MACS)

00378

Term-End Practical Examination

June, 2017

MMT-008(P) : PROBABILITY AND STATISTICS

Time : $1\frac{1}{2}$ Hours *Maximum Marks : 40*

Note : *(i) There are two questions in this paper worth 30 marks. Both the questions are* ***compulsory****.*

(ii) Remaining 10 marks are for viva-voce.

(iii) All the symbols used have their usual meaning.

1. Consider $N_4(\mu, \Sigma)$, where

$$\mu = \begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix} \text{ and } \Sigma = \begin{bmatrix} 7 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 9 \end{bmatrix}.$$

Write a program in 'C' language to obtain the conditional distribution of $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ given $\begin{bmatrix} y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1\cdot2 \\ -2\cdot6 \end{bmatrix}$. *15*

2. Write a program in 'C' language that checks whether a quadratic form in three variables is positive definite or not. It should do the following : *15*

(a) Read the coefficients of the quadratic form.

(b) Print the matrix corresponding to the quadratic form.

(c) Check whether the quadratic form is positive definite or not and print the result.

No. of Printed Pages : 8

MMT-008

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)
M.Sc. (MACS)

00682

Term-End Examination

June, 2017

MMT-008 : PROBABILITY AND STATISTICS

Time : 3 hours *Maximum Marks : 100*

(Weightage : 50%)

Note : *Question no.* ***8*** *is* ***compulsory****. Answer any* ***six*** *questions from questions no. 1 to 7. Use of calculator is* ***not*** *allowed. All the symbols used have their usual meaning.*

1. (a) A cashier in a hospital handles all the payments of the patients. Customers arrive to the cashier at an average of 18 per hour. The service time per customer is, on an average, 2 minutes. Find the following : 5

   (i) Average size of the queue at the cashier window excluding the person getting service.

(ii) Probability that the cashier is idle.

(iii) Average time spent by a new arrival for payment.

(iv) Probability that a customer finds 3 persons ahead of him/her including the one getting service when he arrives.

(b) In a die-coin experiment, a fair die is rolled and then a fair coin is tossed a number of times equal to the score of a die. Find the probability that the coin shows tail at every toss. *6*

(c) A particle moves on a circle through points which have been marked 0, 1, 2, 3, 4 (in a clockwise order). At each step it has a probability p of moving to the right (clockwise) and $1 - p$ to the left (anticlockwise). Let X_n denote its location on the circle after the n^{th} step. The process $\{X_n, n \geq 0\}$ is a Markov chain.

(i) Find the transition probability matrix.

(ii) Calculate the limiting probabilities. *4*

2. (a) The joint probability mass function of random variables X and Y is given in the following table :

y \ x	0	1	2
− 2	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{12}$
0	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{6}$
2	$\frac{1}{6}$	0	$\frac{1}{12}$

(i) Find the marginal distributions of X and Y.

(ii) Find E(X) and E(Y).

(iii) Test the independence of X and Y.

(iv) Find Cov (X, Y). 8

(b) A bivariate population has the following mean vector and variance-covariance matrix :

$$\mu = \begin{bmatrix} 40 \\ 10 \end{bmatrix} \text{ and } \Sigma = \begin{bmatrix} 13 & 10 \\ 10 & 6 \end{bmatrix}.$$

A sample of 10 observations from the population gives sample mean

$$\overline{X} = \begin{bmatrix} 33 \\ 7 \end{bmatrix}.$$

Test whether the sample confirms its truthfulness of mean vector at 5% level of significance. [You may like to use the values, $\chi^2_{2,\,0\cdot05} = 10\cdot6$, $\chi^2_{1,\,0\cdot05} = 7\cdot88$] 7

3. (a) π_1 and π_2 are two populations with probability density functions $p_1(x)$ and $p_2(x)$, respectively. The cost of misclassification of an item in population π_2, given that it was from π_1 is ₹ 100 and the cost of misclassification in population π_1, given that it was from π_2 is ₹ 30. It is known that 70% of the items belong to population π_1.

(i) Find the prior probabilities.

(ii) Find the classification regions.

(iii) If an item has $p_1(x) = 0{\cdot}2$ and $p_2(x) = 0{\cdot}9$, then classify it in π_1 or π_2. 7

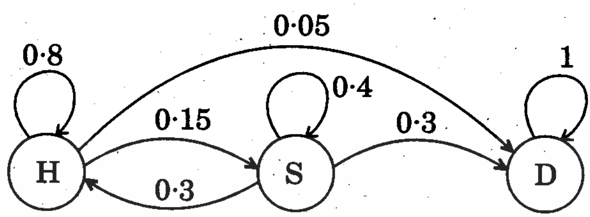
(b) Let $X \sim N_3(\mu, \Sigma)$ where $\mu' = [3, 1, 5]$ and

$$\Sigma = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

(i) Are X_1 and X_3 independent ? Why ?

(ii) Obtain the distribution of CX where $C = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix}$. 8

4. (a) The transition graph of a Markov chain having three states; healthy, sick and dead is given below. The graph given below provides the transition probabilities for the changes in the condition of patients in a hospital in a week.



If the probability that a patient is healthy be 0·7, then find

(i) the probability that the patient is found healthy and will be sick in the coming week.

(ii) the probability that a patient is found healthy and will be sick in the next two weeks.

(iii) Write the transition probability matrix. *7*

(b) Let $\{X_n, n = 1, 2, ... \}$ be i.i.d. geometric variables with probability mass function of each as $p_i = (1-p)\, p^{i-1}, i = 1, 2, ...$.

(i) Find the number of renewals which follow binomial distribution in the corresponding renewal process.

(ii) Find the renewal function of the corresponding renewal process. *4*

(c) If X and Y are two random variables with $E[Y|X] = 1$, show that $\text{Var}(XY) \geq \text{Var}(X)$. 4

5. (a) Find the principal components and proportions of total population variance explained by each component when the covariance matrix is given by $\Sigma = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}$. 7

(b) At 5% level of significance, test $\mu = [7, 11]$ using T^2 from the mean $\overline{X}$ and S^{-1} of a sample of size 10, which are given below :

$$\overline{X} = \begin{pmatrix} 6 \\ 12 \end{pmatrix} \quad S^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{8} \end{bmatrix}.$$

[You may like to use the values

$F_{2, 8, 0\cdot05} = 4\cdot46$, $F_{1, 8, 0\cdot05} = 5\cdot82$] 8

6. (a) In a branching process, offspring distribution follows a geometric distribution as given below :

$$p_k = pq^k;\ q = 1 - p,\ k = 0, 1, 2, \ldots$$

Find the probability of extinction given that

(i) $p = 0\cdot2$

(ii) $p = 0\cdot7$ 7

(b) The transition probability matrix of a Markov chain having three states 1, 2, 3 is given below :

$$\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 1 & 0 \end{bmatrix}$$

(i) Are all states communicable ? Why ?

(ii) Determine the closed set.

(iii) Is the chain irreducible ? Why ?

(iv) Find the probability of ultimate return to the state 1.

(v) Find the mean recurrence time of the state. *8*

7. (a) A drug test is 99% sensitive, that is, positive result is true for drug users and 99% specific, that is, negative result is true for non-drug users. Assume that 1% are drug users. If a randomly selected person is found positive to the test, what is the probability of his/her being a drug user ? *4*

(b) Let $X' = (X_1, X_2, X_3)$ be a random vector and X be the data matrix given below :

$$X = \begin{bmatrix} 5 & 3 & 4 \\ 2 & 1 & 3 \\ 5 & 6 & 4 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

Find

(i) the variance-covariance matrix Σ,

(ii) the correlation matrix R. *8*

(c) Consider the mean vector $\mu_x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mu_y = 2$. The covariance matrices of x_1, x_2 and y are

$$\Sigma_{xx} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \ \sigma_{yy} = 9, \ \sigma_{xy} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Fit the equation $y = b_0 + b_1 x_1 + b_2 x_2$ as best linear equation. *3*

8. State whether the following statements are *True* or *False*. Justify your answers. *5×2=10*

(a) The variance-covariance matrix of a random vector of dimension 2 is given by

$$\begin{bmatrix} 1 & -2 \\ -2 & 2 \end{bmatrix}.$$

(b) If the probability of critical region under null hypothesis H_0 is α, then the probability under alternate hypothesis H_1 is $1 - \alpha$.

(c) A sequence of independent random variables does not form a Markov chain.

(d) The matrix of the quadratic form

$$x_1^2 + 2x_1x_2 + x_2^2 \text{ is } \begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}.$$

(e) For any two events A and B, $P(A) > P(A \mid B)$.

No. of Printed Pages : 1

MMT-008(P)

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE) M.Sc. (MACS)

Term-End Practical Examination

00148

December, 2017

MMT-008(P) : PROBABILITY AND STATISTICS

Time : $1\frac{1}{2}$ Hours *Maximum Marks : 40*

Note : *(i) There are two questions in this paper worth 30 marks. Both the questions are* ***compulsory****.*

(ii) Remaining 10 marks are for the viva-voce.

(iii) All the symbols used have their usual meaning.

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1. Write a program in 'C' language to fit the model $y_i = b_0 + b_1x_{1i} + b_2x_{2i}$, $1 \leq i \leq n$. You may assume that $n \leq 20$. Use the program to fit a linear model for the data given below : 15

y_i	15	25	30	35	40	45	50	55
x_{1i}	5	8	15	20	21	18	10	6
x_{2i}	1	2	3	4	5	3	2	1

2. Write a program in 'C' language to find the multiple correlation coefficient and the mean square error, if Σ_{xx}, σ_{yy} and σ_{xy} are given. 15

No. of Printed Pages : 8

MMT-008

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)
M.Sc. (MACS)

00551

Term-End Examination

December, 2017

MMT-008 : PROBABILITY AND STATISTICS

Time : 3 hours *Maximum Marks : 100*

(Weightage : 50%)

Note : *Question no. 8 is* ***compulsory****. Answer any* ***six*** *questions from questions no. 1 to 7. Use of scientific, non-programmable calculator is allowed. All the symbols used have their usual meaning.*

1. (a) Let X follows $N_3(\mu, \Sigma)$ with

$$\mu = \begin{bmatrix} -3 \\ 1 \\ 4 \end{bmatrix} \quad \Sigma = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix} \text{ and } \mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}.$$

Examine the independence of the following : *8*

(i) X_1 and X_2

(ii) (X_1, X_2) and X_3

(iii) $\dfrac{X_1 + X_2}{2}$ and X_3

(b) On the basis of sales (X_1) and profits (X_2) of 10 industries, the following sample mean was obtained :

$$\overline{X} = \begin{pmatrix} \overline{X}_1 \\ \overline{X}_2 \end{pmatrix} = \begin{pmatrix} 33 \\ 7 \end{pmatrix}$$

Expected mean vector and variance-covariance matrix is

$$\mu = \begin{bmatrix} 30 \\ 10 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 10 & 5 \\ 5 & 4 \end{bmatrix}, \text{ respectively.}$$

Test whether the sample confirms the truthness of mean vector at 5% level of significance. [You may use the values : $\chi^2_{2,\,0\cdot05} = 5\cdot99$, $\chi^2_{9,\,0\cdot05} = 16\cdot92$] 7

2. (a) Let a random variable X have the following probability distribution :

X	− 2	− 1	0	1	2
f(x)	0·15	0·2	0·3	0·2	0·15

Assume $Y = X^2$.

(i) Obtain joint probability distribution of X and Y.

(ii) Obtain covariance of X and Y.

(iii) Examine the independence of X and Y. 7

(b) A post-office has two counters. The first counter handles speed post and the second counter handles all the other tasks. Service times of both the counters follow exponential law with mean 4 minutes. Customers arrive at the first counter at the rate of 12 per hour and at the second counter at the rate of 13 per hour in a Poisson way.

(i) Obtain the average waiting time in queue at both the counters.

(ii) If both the counters are allowed to handle all the tasks, then obtain the average waiting time in queue by the customers. *8*

3\. (a) Obtain a lower triangular square root of the matrix $\begin{bmatrix} 9 & 3 & 3 \\ 3 & 5 & 1 \\ 3 & 1 & 6 \end{bmatrix}$. *6*

(b) Weather data on four variables, x_1 = average minimum temperature, x_2 = average relative humidity in a span of 8 hours, x_3 = average relative humidity in a span of 14 hours and x_4 = total rainfall in 17 years from 1970 to 1986 are provided in the following variance-covariance matrix :

$$\Sigma = \begin{bmatrix} 17{\cdot}02 & -4{\cdot}12 & 1{\cdot}54 & 5{\cdot}14 \\ & 7{\cdot}56 & 8{\cdot}50 & 54{\cdot}82 \\ & & 15{\cdot}75 & 92{\cdot}95 \\ & & & 903{\cdot}87 \end{bmatrix}$$

The eigenvalue-eigenvector pairs of Σ are

$\lambda_1 = 916{\cdot}9$ $\quad a_1 = (0{\cdot}006, 0{\cdot}061, 0{\cdot}103, 0{\cdot}993)$

$\lambda_2 = 18{\cdot}4$ $\quad a_2 = (0{\cdot}955, -0{\cdot}296, 0{\cdot}011, 0{\cdot}012)$

$\lambda_3 = 7{\cdot}9$ $\quad a_3 = (0{\cdot}141, 0{\cdot}485, 0{\cdot}855, -0{\cdot}119)$

$\lambda_4 = 1{\cdot}0$ $\quad a_4 = (0{\cdot}260, 0{\cdot}820, -0{\cdot}509, 0{\cdot}001)$

(i) Obtain principal components.

(ii) Verify that the total variance of principal components is the same as total variance of original variables.

(iii) Obtain proportion of total variation explained by the first component and the first two components.

(iv) Obtain the values of the first two components if the actual data of the first year be $x_1 = 25$, $x_2 = 86$, $x_3 = 66$, $x_4 = 186{\cdot}49$. *9*

4. (a) The P-matrix of a Markov chain is given below. Draw its transition diagram and obtain P^n and its limiting value for large n. *10*

$$P = \begin{bmatrix} 0{\cdot}5 & 0{\cdot}5 & 0 \\ 0{\cdot}25 & 0{\cdot}5 & 0{\cdot}25 \\ 0 & 0{\cdot}5 & 0{\cdot}5 \end{bmatrix}$$

(b) The offspring distribution in a branching process is

$$p_0 = q, p_1 = 0, p_2 = p, p_r = 0, r \geq 3.$$

Discuss the probability of extinction for this process. 5

5. (a) A plant producing memory chips has 3 assembly lines. Line 1 produces 30% of the chips with a defective rate 3%, line 2 produces 50% of the chips with a defective rate 4% and rest of the chips produced by line 3 have a defective rate 2%. A chip is selected from the plant.

(i) Find the probability that the selected chip is defective.

(ii) Given that the chip is defective, find the probability that the chip was produced by line 2. 5

(b) Let Σ be the variance-covariance matrix and **R** be the correlation matrix of a random vector **X**. Let T be a diagonal matrix with the elements being standard deviation of the respective variables. Show that

$$\mathbf{R} = \mathbf{T}^{-1} \Sigma \mathbf{T}.$$ 4

(c) The owner of a chain of five stores wishes to forecast net profit with the help of next year's projected sales of food and non-food items. The data about current year's sales of food items, sales of non-food items as also net profit for all the five stores are available as follows :

Store No.	Net Profit y (₹ in lakhs)	Sales of food items x_1 (₹ in lakhs)	Sales of non-food items x_2 (₹ in lakhs)
1	20	9	6
2	14	7	7
3	7	11	7
4	12	6	10
5	8	5	11

Assuming a linear regression model

$$y_i = b_0 + b_1 x_{1i} + b_2 x_{2i} + e_i,$$

where i = 1, 2, 3, 4, 5.

Find

(i) the least square estimates $\hat{b}$,

(ii) the residuals $\hat{e}$,

(iii) the residual sum of squares for the model. 6

6. (a) Let $X \sim N_3(\mu, \Sigma)$ where $\mu = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ and

$$\Sigma = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}.$$

Obtain the following : 8

(i) Marginal distribution of $\begin{pmatrix} X_1 \\ X_3 \end{pmatrix}$

(ii) Distribution of $Z = X_1 - 2X_2 + X_3$

(iii) Conditional density of $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$, given X_3

(iv) $r_{12 \cdot 3}$

(b) A machine shop needs a certain kind of machine regularly. Whenever a machine fails it is replaced immediately. Assume life time of machines follows uniform distribution in the interval [5, 10] years. Find the rate of replacement in a long time. *4*

(c) If $\{X(t) : t > 0\}$ is a Poisson process with rate λ and S_m denotes the duration from start to the occurrence of m^{th} event, obtain the distribution of S_m. If $\lambda = 1$ per hour, then find the probability that the duration from start to the occurrence of third event will be less than 2 hours. *3*

7. (a) On the basis of 50 observations on 4 variables, the factor loadings of the first two factors obtained through factor analysis are :

Variables	Factors		Communality
	I	II	
X_1	0·697	0·476	0·712
X_2	0·748	0·445	0·758
X_3	0·831	0·350	0·813
X_4	0·596	0·648	0·775
Sum of squares	2·091	0·967	
Variance summarized	0·523	0·242	0·765

(i) Write linear equations for all the factors.

(ii) Interpret the loading coefficients, variance summarized and communality values. *6*

(b) If interoccurrence time in a renewal process follows geometric distribution with parameter p, show that number of occurrences N_n in n time follows binomial distribution. *3*

(c) Let the life times X_1, X_2, ... be i.i.d. exponential random variables with parameter $\lambda > 0$. Let $T > 0$ and age replacement policy is to be employed.

(i) Find mean.

(ii) If each replacement cost $C_1 = 3$ and extra cost $C_2 = 4$, then find the long run average cost per unit time. *6*

8. State whether the following statements are *True* or *False*. Justify your answers. *10*

(a) The probability density function of a random variable lies between 0 and 1.

(b) A state in the Markov chain is transient, if the probability of ultimate return to the state is less than 1.

(c) If $p_0 = 1$ in a branching process, then the probability of ultimate extinction of the process will be smaller than 1.

(d) Posterior probabilities obtained from Bayes' theorem are larger than respective prior probabilities.

(e) T^2 statistics is invariant to the change of origin but not invariant to the change of scale.

No. of Printed Pages : 1

MMT-008(P)

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE) M.Sc. (MACS)

Term-End Practical Examination

June, 2018

00105

MMT-008(P) : PROBABILITY AND STATISTICS

Time : $1\frac{1}{2}$ Hours *Maximum Marks : 40*

Note : *(i) There are two questions in this paper worth 30 marks. Both the questions are **compulsory**.*

(ii) Remaining 10 marks are for viva-voce.

(iii) All the symbols used have their usual meaning.

1. Write a program in C language to fit the model $y_i = b_0 + b_1x_{1i} + b_2x_{2i}$, $1 \le i \le n$. You may assume that $n \le 20$. Use the program to fit a linear model for the data given below :

y_i	12	22	32	40	42	27	18	20
x_{1i}	10	5	7	7	19	22	11	7
x_{2i}	3	4	4	7	7	8	8	9

15

2. Consider $\mathbf{Y} \sim N_5(\mu, \Sigma)$, where

$$\mu = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 4 \\ -3 \end{bmatrix} \quad \text{and} \quad \Sigma = \begin{bmatrix} 5 & 0 & 4 & 2 & 9 \\ 0 & 3 & 2 & 7 & 8 \\ 4 & 2 & 1 & 3 & 5 \\ 2 & 7 & 3 & 2 & 4 \\ 9 & 8 & 5 & 4 & 9 \end{bmatrix}$$

Write a program in 'C' language to obtain the conditional distribution of $\begin{bmatrix} y_1 \\ y_2 \\ y_5 \end{bmatrix}$, given $\begin{bmatrix} y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. 15

No. of Printed Pages : 8

MMT-008

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)
M.Sc. (MACS)

Term-End Examination

00225

June, 2018

MMT-008 : PROBABILITY AND STATISTICS

Time : 3 hours *Maximum Marks : 100*

(Weightage : 50%)

Note : *Question no. 8 is **compulsory**. Answer any **six** questions from questions no. 1 to 7. Use of calculator is allowed. All the symbols used have their usual meaning.*

1. (a) The random variables, X and Y have the following joint p.d.f. :

$$f(x, y) = \begin{cases} x + y, & 0 \le x \le 1, \ 0 \le y \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

Find the

(i) Conditional p.d.f. of X given $Y = 0{\cdot}2$.

(ii) Marginal p.d.f. of X.

(iii) $P[X \le 0{\cdot}4 \mid Y = 0{\cdot}2]$. *6*

(b) A single repairperson looks after two machines, 1 and 2. Each time it is repaired, machine i stays up for an exponential time with rate λ_i, where $i = 1, 2$. When machine i fails, it requires an exponentially distributed amount of work with rate μ_i to complete its repair. The repairperson will always serve machine 1 when it is down. For instance, if machine 1 fails while machine 2 is being repaired, then the repairperson will immediately stop work on machine 2 and start on machine 1.

(i) Write down all the states.

(ii) Find all steady state probabilities.

(iii) What is the probability that the machine 2 is down ? 9

2. (a) Let $Y \sim N_3(\mu, \Sigma)$ where $\mu = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix}$ and

$$\Sigma = \begin{bmatrix} 6 & 1 & 2 \\ 1 & 8 & 4 \\ 2 & 4 & 9 \end{bmatrix}.$$

(i) Obtain distribution of $X = CY$ where $C = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$.

(ii) Obtain $Z = l'Y$ such that $Z \sim N(0, 1)$. 6

(b) Let N_t be a Poisson process with parameter $\lambda > 0$ and $S_{N(t)}$ be the time of last occurrence before time t and $S_{N(t)+1}$ be the time of next occurrence. Show that the residual time $Y = S_{N(t)+1} - t$ follows exponential distribution with parameter λ. *6*

(c) Consider a taxi stand where taxis and customers arrive in accordance with Poisson processes with respective rates of one and two per minute. A taxi will wait no matter how many other taxis are present. However, if an arriving customer does not find a taxi waiting, he leaves.

Find

(i) the average number of taxis waiting, and

(ii) the proportion of arriving customers that get taxis. *3*

3. (a) Let $\overline{X} = (X_1, X_2, X_3)$ be a random vector and data matrix X be given as

$$X' = \begin{bmatrix} 3 & 4 & 2 \\ 5 & 2 & 5 \\ 4 & 2 & 3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}.$$

Obtain

(i) Variance-covariance matrix

(ii) Correlation matrix *10*

(b) In a branching process having offspring distribution,

$$p_j = \begin{cases} \frac{1}{4} & j = 0 \\ \frac{1}{4} & j = 1 \\ \frac{1}{2} & j = 2 \end{cases}$$

find the probability of extinction. *5*

4. (a) In a village 20% children were suffering from malaria. A blood test ordinarily reports positive in 80% cases if the patient was infected by malaria and in 60% cases reports negative if the patient was not infected by malaria. A child was selected at random from the village and the blood test report was positive. Then what is the probability that he is infected by malaria ? *4*

(b) Find the matrix of the following quadratic form and determine its definiteness : *5*

$$2x_1^2 + 3x_2^2 + 4x_3^2 + 6x_1x_2$$

(c) A barber shop has two barbers. Customers arrive at a rate of 5 per hour in a Poisson process and service time of each barber takes on average 15 minutes according to exponential distribution. The shop has 4 chairs for waiting customers. When a customer arrives in the shop and does not find an empty chair, he will leave the shop.

(i) What is the expected number of customers in the shop ?

(ii) What is the probability that a customer will leave the shop finding no empty chair to wait ? 6

5. (a) Evaluate T^2 for testing $H_0 : \mu' = [5, 6]$ using the following data : 9

$$X = \begin{bmatrix} 4 & 5 & 4 & 11 \\ 8 & 7 & 9 & 4 \end{bmatrix}$$

(b) Let π_1 and π_2 be two populations having density functions $P_1(x)$ and $P_2(x)$. Suppose the cost of assigning items to π_1 given π_2 in the true population is 15 and the cost of assigning items to π_2 given π_1 in the true population is 20. It is known that 60% of items belong to π_2.

(i) Find the prior probabilities.

(ii) Write the cost of misclassification.

(iii) Determine classification regions.

(iv) If a new item has $P_1(x) = 0{\cdot}6$ and $P_2(x) = 0{\cdot}4$, then in which population will it be assigned ? 6

6. (a) Let $X \sim N_3(\mu, \Sigma)$ with

$$\Sigma = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Examine the independence of

(i) X_1 and X_2

(ii) (X_1, X_2) and X_3

(iii) $X_1 + X_2$ and X_3 3

(b) The probability transition matrix of a simple weather model having three states, Sunny (S), Cloudy (C) and Rainy (R) with initial probabilities (0·6, 0·3, 0·1) is given below :

$$P = \begin{array}{c} \\ S \\ C \\ R \end{array} \begin{array}{c} \begin{array}{ccc} S & C & R \end{array} \\ \begin{bmatrix} 0{\cdot}5 & 0{\cdot}3 & 0{\cdot}2 \\ 0{\cdot}4 & 0{\cdot}3 & 0{\cdot}3 \\ 0{\cdot}2 & 0{\cdot}5 & 0{\cdot}3 \end{bmatrix} \end{array}$$

(i) Draw the directed graph for the transition matrix.

(ii) Find the probability that three successive days will be sunny starting from the initial day. 6

(c) Let there be three random variables X_1, X_2 and X_3 with the following covariance matrix :

$$\Sigma = \begin{bmatrix} 1 & 0{\cdot}6 & 0{\cdot}4 \\ 0{\cdot}6 & 1 & 0{\cdot}3 \\ 0{\cdot}4 & 0{\cdot}3 & 1 \end{bmatrix}$$

Take one underlying factor and write its factor model. 6

7. (a) Determine the nature of states from the following transition matrix of a Markov chain : 8

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0{\cdot}5 & 0{\cdot}5 & 0 \\ 0{\cdot}75 & 0 & 0{\cdot}25 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

(b) For a Poisson process $\{X(t) : t \geq 0\}$, define processes $N(t) = X(t + s_0) - X(t)$ and $U(t) = X(t + s_0) - X(s_0)$. Examine which of the processes N(t) and U(t) is a Poisson process. 7

8. State whether the following statements are *True* or *False*. Justify your answers. 5×2=10

(a) Transition probabilities $p_{ij}^{(m, n)}$ in a time homogeneous Markov chain depend on specific times m, n.

(b) In a branching process, if the offspring probability $p_0 = 0$, then there will never occur an extinction of the process.

(c) If $P(B) > 0$, then $P(A|B)$ is smaller than $P(A \cap B)$.

(d) If A is a positive definite matrix, then $|A| > 0$.

(e) The principal components of a set of variables do not depend upon the scales of the variables.

189201

No. of Printed Pages : 2

MMT-008(P)

M. Sc. (Mathematics with Applications in Computer Science) M. Sc. (MACS)

Term-End Examination

December, 2018

PROBABILITY AND STATISTICS

Time : $1\frac{1}{2}$ Hours *Maximum Marks : 40*

Note : (i) There are ***two*** questions in this paper worth 30 marks. Both questions are compulsory.

(ii) Remaining 10 marks are for viva-voce.

(iii) All the symbols used have their usual meaning.

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1. Write a program in 'C' language that checks whether a Var-Cov matrix is positive definite or not. 10

2. Consider $\underset{\sim}{Y} \sim N_3(\underset{\sim}{\mu}, \underset{\sim}{\Sigma})$, where :

$$\underset{\sim}{\mu} = \begin{bmatrix} 3 \\ 4 \\ -5 \end{bmatrix} \text{ and } \underset{\sim}{\Sigma} = \begin{bmatrix} 2 & -2 & -1 \\ -2 & 5 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Write a program in 'C' language to find the distribution of $\underset{\sim}{C}\,\underset{\sim}{Y}$, where $\underset{\sim}{C}$ is any matrix of order 2×3. Test your program for : 20

$$\underset{\sim}{C} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

No. of Printed Pages : 11

MMT-008

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE) M.Sc. (MACS)

Term-End Examination

00942

December, 2018

MMT-008 : PROBABILITY AND STATISTICS

Time : 3 hours *Maximum Marks : 100*

(Weightage : 50%)

Note : *Question no. 8 is **compulsory**. Answer any **six** questions from questions no. 1 to 7. Use of calculator is **not** allowed. All the symbols used have their usual meaning.*

1. (a) The joint probability mass function of two random variables X and Y is given in the following table :

X \ Y	−1	0	1
1	0·2	0·2	0·2
2	0·2	0·1	0·1

(i) Find E(X), E(Y), V(X) and V(Y).

(ii) Test independence of X and Y.

(iii) Find $E(X \mid Y = 0)$ and $V(Y \mid X = 1)$.

(iv) Obtain Cov (X, Y). *10*

(b) Suppose that families migrate to an area at a Poisson rate of $\lambda = 2$ per week. If the number of people in each family is independent and takes on the values 1, 2, 3, 4 with respective probabilities $\frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{6}$, then find the expected value and variance of the number of individuals migrating to this area during a fixed five-week period. *5*

2\. (a) Three players A, B, C in a circle, throw a ball to the left player with probability 0·4 and to the right player with probability 0·6.

(i) Write the transition probability matrix.

(ii) Find $P^{(2)}$.

(iii) The probability that the ball is with any one player is the same, i.e. 1/3 at the start. What will be the probability that after two throws the ball will be with A ? 6

(b) Consider a three-state Markov chain having the following transition probability matrix :

$$P = \begin{bmatrix} 0{\cdot}5 & 0{\cdot}4 & 0{\cdot}1 \\ 0{\cdot}3 & 0{\cdot}4 & 0{\cdot}3 \\ 0{\cdot}2 & 0{\cdot}3 & 0{\cdot}5 \end{bmatrix}$$

Find the limiting probabilities of all the three states. 4

(c) A supermarket has two exponential checkout counters, each operating at the rate μ. Arrivals are Poisson at the rate λ. The counters operate in the following way :

- One queue feeds both counters.
- One counter is operated by a permanent checker and the other by a stock clerk, who instantaneously begins checking whenever there are two or more customers in the system. The clerk returns to stocking whenever service is complete, and there are fewer than two customers in the system.

(i) Let P_n be the proportion of time when there are n customers in the system. Set up equations for P_n and solve them.

(ii) At what rate does the number in the system go from 0 to 1 and from 1 to 2 ?

(iii) What proportion of time is the stock clerk checking ? 5

3. (a) Let $\mathbf{y} \sim N_3(\boldsymbol{\mu}, \Sigma)$, where $\boldsymbol{\mu} = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$ and

$\Sigma = \begin{bmatrix} 6 & 1 & 2 \\ 1 & 8 & 4 \\ 2 & 4 & 9 \end{bmatrix}$. Obtain the distribution of

$\mathbf{Z} = \mathbf{C}\mathbf{y}$, where $\mathbf{C} = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 8 & 1 \end{bmatrix}$. 5

(b) Suppose in a city 20% individuals are smokers. It is also observed that 70% of smokers and 30% of non-smokers are males. An individual is chosen randomly from the city. What is the probability that the person is a male ? If it is observed that the person is a male, then what is the probability that he is a smoker ? 4

(c) Customers arrive at a counter with Poisson rate 8 per hour. Service time follows exponential distribution with mean 5 minutes. Find

(i) the probability that a customer will have to wait before service;

(ii) the proportion of time the counter is idle;

(iii) the probability that the total time spent at the counter is more than 10 minutes;

(iv) the average waiting time at the counter. 6

4. (a) Let **X** = X_1, X_2, X_3 be a random vector and let the data matrix for X be

$$\begin{bmatrix} 5 & 2 & 5 \\ 3 & 4 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

Obtain

(i) the variance-covariance matrix,

(ii) the correlation matrix. 9

(b) The distribution of a geometric random variable describing the offspring in a branching process is given by

$p_k = pq^k$, $q = 1 - p$, $0 < p < 1$, $k = 0, 1, 2, \ldots$.

Find the probability of extinction of the process when (i) $p = 0{\cdot}2$, (ii) $p = 0{\cdot}6$. *6*

5. (a) The variables X_1, X_2 and X_3 have the following variance-covariance matrix :

$$\Sigma = \begin{bmatrix} 1 & 0{\cdot}63 & 0{\cdot}45 \\ 0{\cdot}63 & 1 & 0{\cdot}35 \\ 0{\cdot}45 & 0{\cdot}35 & 1 \end{bmatrix}.$$

Write its factor model. *6*

(b) A Markov chain has the following transition matrix :

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0{\cdot}5 & 0{\cdot}5 & 0 \\ 0{\cdot}75 & 0 & 0{\cdot}25 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

Determine the probabilities of ultimate return to the states and mean recurrence times of the states. Is the chain irreducible ? Give reasons for your answer. *6*

(c) Three white and three black balls are distributed in two urns in such a way that each contains three balls. The system is said to be in state i, $i = 0, 1, 2, 3$, if the first urn contains i white balls. At each step, one ball is drawn from each urn and the ball drawn from the first urn is placed into the second, and conversely the ball drawn from the second urn is placed in the first urn. Let X_n denote the state of the system after the n^{th} step. Check whether $\{X_n, n = 0, 1, 2, ...\}$ is a Markov chain or not. If yes, write its transition probability matrix. If it is not a Markov chain, give the constraints so that it becomes Markovian. *3*

6. (a) Sales X_1 and profits X_2 of an industry have the following population mean and var-cov matrices

$$\mu = \begin{bmatrix} 30 \\ 10 \end{bmatrix}, \ \Sigma = \begin{bmatrix} 10 & 5 \\ 5 & 4 \end{bmatrix}.$$

A sample of 10 industries gave sample mean $\bar{X} = \begin{bmatrix} 33 \\ 7 \end{bmatrix}$. Test, at 5% level of significance, for the truthfulness of the population mean.

[You may like to use the following values :

$\chi^2_{2,0\cdot05} = 5\cdot99$, $\chi^2_{3,0\cdot05} = 7\cdot81$,

$\chi^2_{4,0\cdot05} = 9\cdot48$]. 8

(b) $\{X(t),\ t > 0\}$ is a Poisson process with parameter λ and δ_m denotes the duration from the beginning to the occurrence of the m^{th} event. Obtain the distribution of δ_m. Also, find cdf of δ_m. If $\lambda = 1$ per hour, then find the probability that duration from the start to the occurrence of the third event will be less than 2 hours. 7

7. (a) Consider the mean vectors $\mu_X = [1, 2]'$ and $\mu_Y = 3$. The var-cov matrix of $[X_1, X_2]'$ is $\Sigma_{XX} = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ and, $\sigma_{YY} = 14$, $\sigma_{XY} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(i) Fit the equation $Y = b_0 + b_1X_1 + b_2X_2$ as the best linear equation.

(ii) Find the multiple correlation coefficient.

(iii) Find the mean square error. 8

(b) Let 10 and 15 observations be taken for the random variables X_1 and X_2 from the populations π_1 and π_2 respectively. Let $\mu^{(1)} = [3, 1]'$, $\mu^{(2)} = [2, 1]'$ and $\Sigma^{-1} = \begin{bmatrix} 0{\cdot}15 & 0{\cdot}05 \\ 0{\cdot}05 & 0{\cdot}15 \end{bmatrix}$. Assuming equal cost and equal prior probabilities, check whether the observation $[3, 1]'$ belongs to the population π_1 or π_2. 7

8. State whether the following statements are *True* or *False*. Justify your answers with a short proof or a counter example. *5×2=10*

(a) If the transition matrix of a stochastic process is $P = \begin{bmatrix} 0{\cdot}3 & 0{\cdot}7 \\ 0{\cdot}7 & 0{\cdot}3 \end{bmatrix}$, then $\lim_{n \to \infty} P^n = \begin{bmatrix} 0{\cdot}5 & 0{\cdot}5 \\ 0{\cdot}5 & 0{\cdot}5 \end{bmatrix}$.

(b) In a three-state Markov chain, all the states can be transient.

(c) If in a queuing system M/M/1, $L_s = 10$ and $W_s = 5$ minutes, then the arrival rate will be 3 per minute.

(d) The matrix $\begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}$ is a variance-covariance matrix of two random variables.

(e) In the individual replacement policy, the components are replaced at fixed time periods T, 2T, 3T, ...

01131

No. of Printed Pages : 8

MMT-008

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE) M.Sc. (MACS)

Term-End Examination, 2019

MMT-008 : PROBABILITY AND STATISTICS

Time : 3 Hours] [Maximum Marks : 100

(Weightage : 50%)

Note : Question No. 8 is compulsory. Attempt **any six** questions from question no. 1 to 7. Use of calculator is not allowed. All the symbols used have their usual meaning.

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1. (a) Let the joint probability density function of two continuous random variables X and Y be

$$f(x, y) = 8xy, 0 < x < y < 1$$
$$= 0 \quad , \text{elsewhere}$$

(i) Find the marginal p.d.f. of X and Y.

(ii) Test independence of X and Y.

(iii) Compute $P[0 < x < 0.4 \mid 0.3 < y < 0.8]$

(iv) Find $V(Y \mid X = x)$. [9]

(b) Let $X \sim N_3(\mu, \Sigma)$ where $\mu = [213]'$ and $\Sigma = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$. Find the distribution of

$$\begin{bmatrix} X_1 - X_2 + X_3 \\ X_1 + X_2 + 2X_3 \end{bmatrix}.$$ [6]

2. (a) A Markor chain $\{X_n, n=0,1, 2, \ldots\}$ has initial distribution $u_0 = [0.1, 0.3, 0.6]'$ and transition matrix $P = \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.2 & 0.5 \\ 0.3 & 0.3 & 0.4 \end{bmatrix}$ having states (1,2,3), obtain :

(i) $P[X_2 = 3]$

(ii) $P[X_1 = 2, X_2 = 3]$

(iii) $P[X_0 = 1, X_1 = 3, X_2 = 2]$ [6]

(b) Determine the principal components, Y_1 and Y_2, for the covariance matrix $\Sigma = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}$. Also calculate the proportion of total population variance for the first principal component. [9]

3. (a) The mean Poisson rate of arrival of planes at an airport during peak hours is 20 per hour. 60 planes per hour can land at the airport in good weather and 30 planes per hour in bad weather in Poisson fashion. Find the following during peak hours :

(i) The average number of planes flying over the field in good weather ;

(ii) The average number of planes flying over the field in bad weather ;

(iii) The average number of planes flying over the field and landing in good weather ;

(iv) The average number of planes flying over the field and landing in bad weather ;

(v) The average landing time in good weather and bad weather. [6]

(b) An equal number of balls are kept in three boxes B_1 , B_2 and B_3. The boxes B_1 , B_2 and B_3 contain respectively 3%, 5% and 2% defective balls. One of the boxes is selected at random and a ball is drawn randomly. If the ball is found to be defective, what is the probability that it has come from B_2? [4]

(c) Let $Z = [Z^{(1)}, Z^{(2)}]$ and

$$\text{Cov}(Z) = \left[\begin{array}{c|c} \rho_{11} & \rho_{12} \\ \hline \rho_{21} & \rho_{22} \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0.4 & 0.5 & 0.6 \\ 0.4 & 1 & 0.3 & 0.4 \\ \hline 0.5 & 0.3 & 1 & 0.2 \\ 0.6 & 0.4 & 0.2 & 1 \end{array}\right]$$

Compute the correlation between the first pair of canonical variates and their component variables. [5]

4. (a) From the samples of sizes 80 and 100 from two populations, the following summary statistics were obtained :

$$X_1 = \begin{bmatrix} 8 \\ 4 \end{bmatrix}, X_2 = \begin{bmatrix} 10 \\ 4 \end{bmatrix}, S_1 = \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}, S_2 = \begin{bmatrix} 2 & 1 \\ 1 & 6 \end{bmatrix}$$

Where X_1, X_2 are the means and S_1, S_2 are the standard deviations of two populations. Test for the equality of the population means at 5% level of significance. Assume $\sum_1 = \sum_2$. [You may use the following values : $F_{0.05,2,177} = 3.04$, $F_{0.05,2,100} = 3.10$, $F_{0.05,2,80} = 3.15$]. [7]

(b) Describe birth and death processes with the parameter λ. If $\lambda_k = \lambda$ and $\mu_k = k\mu$, $k \geq 0$, λ, $\mu > 0$, then show that the stationary distribution of these process always exists. Obtain the steady state distribution. [8]

5. (a) Let X be $N_3(\mu, \Sigma)$ with $\Sigma = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Examine the independence of the following :

(i) X_1 and X_2 ;

(ii) (X_1, X_2) and X_3 ;

(iii) $X_1 + X_2$ and X_3. [7]

(b) Let { N_n, n= 0,1,2, ...} be a renewal process with sequence of renewal periods $\{X_i\}$. Each X_i follows the binomial distribution with $P[X_i = 0] = 0.6$ and $P[X_i = 1] = 0.4$.

Find the distribution of N_n. [5]

(c) To fit linear are regression on dependent variable Y and independent variables X_1 and X_2 we have the following information :

$E(X_1) = 3$, $E(X_2) = 2$, $\text{Var}(X_1) = 2$, $\text{Var}(X_2) = 1$, $\text{Cov}(X_1, X_2) = 1$, $\text{Cov}(X_1, Y) = 3$, $\text{Cov}(X_2, Y) = 1$, $V(Y) = 9$. Find the multiple correlation coefficient R. [3]

6. (a) A barber shop has two barbers. The customers arrive at a rate of 5 per hour in a Poisson fashion, and the service time of each barber takes an average of 15 minutes according to exponential distribution. The shop has 4 chairs for waiting customers. When a customer arrives in the shop and does not find an empty chair, she leaves the shop. What is the expected number of customers in the shop ? What is the probability that a customer will leave the shop finding no empty chair to wait ? [5]

(b) In a branching process, the offspring distribution (p_k) is given below :

$$p_k = pq^k, q = 1 - p, 0 < p < 1, k = 0, 1, 2, \ldots\ldots$$

What will be the probability of extinction in this branching process ? [5]

(c) Let N_s be a Poisson process with parameter $\lambda > 0$. Fix $s > 0$ and let the renewal function be given by $M_t = N_{(t+s)} - N_s$. Show that the conditional distribution of M_t, given $N_s = 10$, is Poisson. [5]

7. (a) Suppose $n_1 = 20$ and $n_2 = 30$ observations are made on two variables X_1 and X_2 where $X_1 \sim N_2(\mu^{(1)}, \sum)$ and $X_2 \sim N_2(\mu^{(2)}, \sum)$

$$\mu^{(1)} = [1\ 2]' \quad , \quad \mu^{(2)} = [-1\ 0]' \quad \sum = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}.$$

Considering equal cost and equal prior probabilities, classify the observation [-1 1]' in one of the two populations. [6]

(b) Suppose interoccurrance times $\{X_n : n \geq 1\}$ are uniformily distributed on $[0,1]$:

(i) Find $\overline{M}_t$, the Laplace transform of the renewal function M_t.

(ii) Find $\lim_{t \to \infty} M_t/t$. [4]

(c) Let the rand m vector $X' = [X_1\ X_2\ X_3]$ has

Mean Vector $= \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$ and

Var-cov matrix $= \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \\ 3 & 1 & 9 \end{bmatrix}$. Fit the equation

$Y = b_0 + b_1 X_1 + b_2 X_2$. Also obtain the multiple correlation coefficient between X_3 and $[X_1, X_2]$.[5]

8. State whether the following statements are true or false. Justify your answer with a short proof or a counter example : [10]

(i) If P is a transition matrix of a Markov Chain, then all the rows of $\lim\limits_{n \to \infty} P^n$ are identical.

(ii) Every non-negative definite matrix is a var-cov matrix.

(iii) The multiple correlation coefficient R can lie between -1 and 0.

(iv) Posterior probabilities obtained from Baye's theorem are larger than respective prior probabilities.

(v) If X_1, X_2, X_3 are iid from $N_2(\mu, \Sigma)$, then $\dfrac{X_1 + X_2 + X_3}{3}$ follows $N_2(\mu, \frac{1}{3}\Sigma)$.

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Number of Printed Pages : 9

MMT-008

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

M.Sc. (MACS)

Term-End Examination, 2019

MMT-008: PROBABILITY AND STATISTICS

Time : 3 Hours

Maximum Marks : 100

(Weightage : 50%)

Note : **Question No.8** is **compulsory**. Attempt any six questions from question no. 1 to 7. Use of scientific non-programmable calculator is allowed. Symbols have their usual meanings.

1. (a) In a dice-coin experiment, a fair dice is rolled and then a fair coin is tossed a number of times equal to the score on the dice: [6]

   (i) Find the probability that the coin shows tail in every toss.

(ii) Find the probability that the score of the dice was 2, given that the coin showed tail in all tosses.

(b) For 10 pairs of observations from a bivariate normal population, the following statistics were obtained : [9]

$$\overline{X} = \begin{bmatrix} 3.5 \\ 7 \end{bmatrix}, S = \begin{bmatrix} 0.4 & 0.2 \\ 0.2 & 0.6 \end{bmatrix}$$

Test the hypothesis $\overline{\mu}' = [3, \ 8]$ at 5% level of significance.

[You may like to use the following values : $F_{2,8}(0.05) = 13.23$, $F_{2,9}(0.05) = 15.28$, $F_{2,10}(0.05) = 17.39$]

2. (a) Given a Markov chain having following transition matrix : [8]

$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 0 \\ 1 & 1/3 & 0 & 2/3 \\ 2 & 0 & 1 & 0 \end{array}$$

(i) Check whether the chain is irreducible.

(ii) Find the probabilities of ultimate return to the states.

(iii) Find the mean recurrence time to the states.

(b) Let $\underset{\sim}{y} \sim N_4(\underset{\sim}{\mu}, \underset{\sim}{\Sigma})$ where $\underset{\sim}{\mu}' = [-2\ 3\ 0\ 5]$ and [7]

$$\underset{\sim}{\Sigma} = \begin{bmatrix} 4 & -2 & 0 & 3 \\ -2 & 7 & 0 & -1 \\ 0 & 0 & 3 & 1 \\ 3 & -1 & 1 & 5 \end{bmatrix}.$$

Obtain :

(i) Marginal distribution of $\begin{bmatrix} y_1 \\ y_3 \end{bmatrix}$.

(ii) Conditional distribution of $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ when $\begin{bmatrix} y_3 \\ y_4 \end{bmatrix}$ is given.

(iii) Correlation coefficient r_{12} and $r_{12.34}$.

3. (a) The variance-covariance matrix of three variables x_1, x_2 and x_3 is : [10]

$$\underset{\sim}{\Sigma} = \begin{bmatrix} 109 & 198 & 110 \\ 198 & 418 & 197 \\ 110 & 197 & 107 \end{bmatrix}$$

The eigen values and the corresponding eigen vectors are given below :

$\lambda_1 = 607.2$ $\quad \underset{\sim}{a}_1' = [0.80, \; 0.42, \; 0.80]$

$\lambda_2 = 26.4$ $\quad \underset{\sim}{a}_2' = [0.92, \; -0.97, \; 0.23]$

$\lambda_3 = 5.4$ $\quad \underset{\sim}{a}_3' = [-0.35, \; 0.08, \; 0.86]$

(i) Obtain principal components.

(ii) Obtain variances of principal components.

(iii) Check whether the total variance explained by the principal components is equal to the total variance of the original variables.

(iv) Obtain the proportions of variation explained by first two components.

(b) Define a renewal process with example. Check whether a poisson process is a renewal process or not. If the renewal periods in a renewal process are iid exponential, then find the distribution of renewal sequence. [5]

4. (a) Let $\underset{\sim}{y} \sim N_3(\underset{\sim}{\mu}, \underset{\sim}{\Sigma})$ where : [7]

$$\underset{\sim}{\mu} = \begin{bmatrix} 4 \\ 6 \\ -1 \end{bmatrix}, \quad \underset{\sim}{\Sigma} = \begin{bmatrix} 4 & 4 & 1 \\ 4 & 7 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

Obtain :

(i) The distribution of $\underset{\sim}{c}\,\underset{\sim}{y}$, where

$$\underset{\sim}{c} = \begin{bmatrix} 4 & 1 & 5 \\ 5 & 4 & 3 \end{bmatrix}.$$

(ii) The linear combination $\underset{\sim}{z} = [1 \quad 1 \quad 1]\,\underset{\sim}{y}$ such that $\underset{\sim}{z} \sim N\,(0, 1)$.

(b) One of the two teller machine handles withdrawals and the other handles deposits in a bank. The service time of both the machines follows exponential distribution with mean service time 3 minutes. The depositors arrive in the bank at the rate of 16 per hour and withdrawers arrive at the rate of 14 per hour in Possion distribution.

Find the average waiting time of depositors and withdrawers in the queue. If each machine can handle both the jobs of deposit and withdrawal, then what will be the average waiting time in the queue for a customer ? [8]

5. (a) Consider a branching process with offspring distribution given by : [6]

$$p_j = \begin{cases} 2/7 & , \quad j = 0 \\ 5/7 & , \quad j = 2 \end{cases}$$

find the probability of extinction

(b) The variance-covariance matrix of three random variables X_1, X_2 and X_3 is given by : [9]

$$\underset{\sim}{\Sigma} = \begin{bmatrix} 1 & 0.36 & 0.49 \\ 0.36 & 1 & 0.25 \\ 0.49 & 0.25 & 1 \end{bmatrix}$$

Write its factor model.

6. (a) Consider a 3-state Markov Chain with the following transition matrix : [5]

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 2/5 & 1/5 & 2/5 \\ 3/7 & 2/7 & 2/7 \end{bmatrix}$$

Find the stationary distribution of P.

(b) Two random variables x and y have the following joint pdf : [6]

$$f(x,y) = \frac{3}{16}(x^2 + y^2);\ 0 < y < x \le 2$$

(i) Find the marginal distribution of y.

(ii) Compute the conditional expectation of y given x.

(iii) find $P\left[\frac{1}{2} < y < 1 / \frac{1}{2} < x < \frac{3}{2}\right]$

(c) Derive Chapman-Kolomogorov equations by stating all the assumptions. [4]

7. (a) The transition probability matrix P of a Markov Chain with states sunny (S), cloudy (C) and rainy (R) in a simple weather model is given by as :

$$P = \begin{array}{c} \\ S \\ C \\ R \end{array} \begin{array}{c} \begin{array}{ccc} S & C & R \end{array} \\ \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.1 & 0.2 & 0.7 \end{bmatrix} \end{array}$$

The initial distribution of the states is [0.5, 0.3, 0.2]. [5]

(i) What is the probability that starting from initial day all the three successive days will be cloudy ?

(ii) Obtain the probability distribution of the weather on second day.

(b) A radioactive source emits particles at a rate of 5 per minute according to Poisson law. Each particle emitted has probability 0.6 of being recorded. What is the probability that in 4 minutes 10 particles will be recorded ? Also, find the mean and variance of the number of particles recorded. [5]

(c) Let the lifetimes $X_1, X_2, \ldots\ldots\ldots\ldots$ of a process be iid exponential random variables with parameters $\lambda > 0$. Let T > 0 and age replacement policy be employed : [5]

(i) Find the mean of the replacement time.

(ii) If replacement cost is 3 each time and the extra cost is 4, then find the long run average cost per unit time.

8. Which of the following statements are true and which are false ? Give a short proof or a counter example in support of your answer : [10]

(a) The matrix with sum of principal diagonal elements as 1 is a transitional probability matrix.

(b) If $A \subset B$, then $P(A | B) \geq P(A)$.

(c) Every variance-covariance matrix is a non-negative definite matrix.

(d) In a linear regression model, the extent of fit is measured by partial correlation coefficient.

(e) In a renewal process, $P[N(t) \geq n] > P[S_n \leq t]$.

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MMT-008

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE

Term-End Examination

MMT-008 : PROBABILITY AND STATISTICS

Time : 3 Hours] *[Maximum Marks : 100*

Note: 1. Q. No. 8 is compulsory.

2. Attempt any six questions from Q No. 1 to 7.
3. Use scientific non prgramable calculators is allowed.
4. Symbols have their usual meanings.

1. (a) Consider the Markov chain having the following transition probability matrix : 10

$$P = \begin{bmatrix} 1/3 & 0 & 2/3 & 0 & 0 & 0 \\ 0 & 1/2 & 1/4 & 0 & 1/4 & 0 \\ 2/5 & 0 & 3/5 & 0 & 0 & 0 \\ 0 & 1/4 & 1/4 & 1/4 & 0 & 1/4 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 1/4 & 3/4 \end{bmatrix}$$

(i) Draw the digram of the Markov chain.

(ii) Write the classes of persistent, non-null and aperiodic states.

No. of Printed Pages : 8

MMT-008

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE

Term-End Examination

MMT-008 : PROBABILITY AND STATISTICS

Time : 3 Hours] *[Maximum Marks : 100*

Note: 1. Q. No. 8 is compulsory.

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4. Symbols have their usual meanings.

1. (a) Consider the Markov chain having the following transition probability matrix : 10

$$P = \begin{bmatrix} 1/3 & 0 & 2/3 & 0 & 0 & 0 \\ 0 & 1/2 & 1/4 & 0 & 1/4 & 0 \\ 2/5 & 0 & 3/5 & 0 & 0 & 0 \\ 0 & 1/4 & 1/4 & 1/4 & 0 & 1/4 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 1/4 & 3/4 \end{bmatrix}$$

(i) Draw the digram of the Markov chain.

(ii) Write the classes of persistent, non-null and aperiodic states.

(iii) Find the probability of absorption to the classes. Also, find the mean time upto absorption from transient state 2 to 4.

(b) Find the principal components and proportions of total population variance explained by each component when the covariance matrix is.

$$\Sigma = \begin{bmatrix} 5 & 2 \\ 2 & 7 \end{bmatrix}$$ 5

2. (a) A communication system transmits two digits 0 and 1, each of them passing through several stages. Suppose the probability at the time of leaving of the digit that enters remain unchanged is a p and when it changes is 1-p. Suppose that X_0 is the digit that enters in the first stage of the system and X_n $(n \leq 1)$ is the digit O leaving in the nth stage of the system. Show that $\{x_n, n \geq 1\}$ forms a Markov chain. Find P, P², P³ and compute $P\left[X_2 = 0 | Xo = 1\right]$ and $P\left[X_3 = 1 | Xo = 2\right]$.

Let $\underset{\sim}{X}_n N_4(\underset{\sim}{u}, \underset{\sim}{\Sigma})$ with

$$\underset{\sim}{u} = \begin{bmatrix} 2 \\ 4 \\ -1 \\ 3 \end{bmatrix} \quad \text{and} \quad \underset{\sim}{\Sigma} = \begin{bmatrix} 4 & 2 & -3 & 4 \\ 2 & 4 & 0 & 1 \\ -3 & 0 & 4 & -2 \\ 4 & 1 & -2 & 8 \end{bmatrix}.$$

Suppose $\underset{\sim}{Y}$ and $\underset{\sim}{Z}$ are two partitioned sub-vectors of $\underset{\sim}{X}$ such that $\underset{\sim}{Y}' = [x_1 x_2]$ and $\underset{\sim}{Z}' = [x_3 x_4]$ find. 7

(i) E (y/z)

(ii) Cov. (y/z)

(iii) Correlation coefficient $r_{12.34}$. 8

3. (a) The number of accidents in a town follows a poisson process with a mean of 2 per day and the number X_i of people involved in the ith accident be the iid given as

$P[xi = p] = \frac{1}{2^k}; k \geq 1.$ Find the mean and variance of the number of people involved in accident per week. 5

(b) Suppose 10 and 15 observations are made on two random variables $\underset{\sim}{X}_1 \sim N_2(\underset{\sim}{u}_1, \underset{\sim}{\Sigma})$ and $\underset{\sim}{X}_2 \sim N_2(\underset{\sim}{u}_2, \underset{\sim}{\Sigma})$, where $\underset{\sim}{lu} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \underset{\sim}{u}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and

$\underset{\sim}{\Sigma} = \begin{bmatrix} 1.5 & 2 \\ 2 & 4 \end{bmatrix}$. Considering equal cost and equal prior probabilities, classify the observation $[1.5, 1]$ in one of the two populations. 10

4. (a) Find the differential equation of pure death process. If the process Narts with i individuals, find the mean and variance of the number N(t) present at time t. 6

(b) Define conjoint analysis with suitable example. How conjoint analysis is used to optimize product design? 5

(c) The inter occurence time in a renewal process follower exponential distribution with rate $\lambda > 0$. What will be the distribution of number of renewals in time t? Obtain renewal fucntion and its a place transform. 4

5. (a) Patients arrive at the outpatient department of a hospital in accordance with a poisson process at the mean rate of 12 per hour, and the distribution of time for examination by an attending physician is exponential with mean of 10 minutes. What is the minimum number of physicians to be posted for ensuring a steady state distribution. For this number, find.

(i) The expected waiting time of a patient prior to being examined.

(ii) The expected number of patients in the out-patient department. Also, find the average number of physicians who remain idle.

(b) Find all stationary distributions for a Markov chain having the following tarnsition probability matrix : $P = \begin{bmatrix} 0.4 & 0.6 \\ 0.7 & 0.3 \end{bmatrix}$ 6

6. (a) Suppose that each individuatl at the end of the unit of time produces either k (k >, 2) or O direct descendants with probabilities p or q, respectively. Check whether the probability of ultimate extinction is less than 1. Find this probability for K=3 when þ=q=½. 6

(b) Let $\underset{\sim}{X} \sim N_3(u, \Sigma)$ with the date matrix

$$X = \begin{bmatrix} 5 & 3 & 4 \\ 2 & 1 & 3 \\ 5 & 6 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$ Find the variance-covariance matrix $\underset{\sim}{\Sigma}$ and the correlation matrix $\underset{\sim}{R}$. 9

7. (a) The joint probability mass function of random variables X and Y is given by :

$y \backslash x$	0	1	2
-2	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{12}$
0	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{6}$
0	$\frac{1}{6}$	0	$\frac{1}{12}$

(i) Find the marginal distributions of X and Y.

(ii) Find E (X) and E (Y).

(iii) Find Cove (X,V).

(b) Consider M/M/1 queveing system with arriva rate λ and service rate 2 λ and another M M/2 queveing system with arrival rate λ a service rat λ. Show that the average wait time in system M/M/1 is smaller than waiting time in M/M/2 system.

(c) Define canonical correlation with suitai example. How canonical correlation is us to do optional scaling.

8. State which of the following statements are tr and which are false. Give a short proof or counter example in support of your answer. 1

(i) Two independent events B and C are such that $P(B \cap C) > 0$, then $P(A / B \cap C) = P(A / B).P(A / C)$.

(ii) If X and Y are two random variables with V(X)=V(Y)=2, then –2<Cov (X,Y) <2.

(iii) The row sums in the infinitesimal generator of a birth and death process are zero.

(iv) A real symmetric matrix $(a_{ij})_{n \times n}$ with a_{ii}=-1 cannot be positive definite.

(v) The maximum likelihood estimator of $u' \Sigma^{-1} u$ is $\bar{X}' U^{-1} \bar{X}$, where $\bar{X}$ and U are the maximum likelihood estimators of u and Σ respectively.

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MMT-008

M.Sc. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE)

M.Sc. (MACS)

Term-End Examination

February, 2021

MMT-008 : PROBABILITY AND STATISTICS

Time : 3 hours | *Maximum Marks : 100*
(Weightage : 50%)

Note : *Question no. 8 is* ***compulsory****. Answer any* ***six*** *questions from question nos. 1 to 7. Use of non-programmable calculator is* ***allowed****. All the symbols used have their usual meaning.*

1. (a) Let the joint probability density function of two discrete random variables X and Y be given as : 8

		X			
		2	3	4	5
Y	0	0	0·03	0	0
	1	0·34	0·30	0·16	0
	2	0	0	0·03	0·14

(i) Find the marginal distribution of X and Y.

(ii) Find the conditional distribution of X given Y = 1.

(iii) Test the independence of variables X and Y.

(iv) Find $V\left[\frac{Y}{X} = x\right]$.

(b) Let $p_k = bc^{k-1}$, $k = 1, 2, ..., 0 < b, c, \; b + c < 1$,

and $p_0 = 1 - \sum_{k=1}^{\infty} p_k$.

Find the probability of ultimate extinction. *7*

2. (a) A computer lab has a help desk to assist students working on computer spreadsheet assignments. The students form a single line and are served on a first-come first-served basis. On average, 15 students per hour arrive in a Poisson distribution. The help desk can help an average of 20 students per hour and the service rate is exponentially distributed.

Find

(i) The average utilization of the help desk.

(ii) The average number of students in the system.

(iii) The average number of students waiting in line.

(iv) The average time a student spends in the system.

(v) The average time a student spends waiting in line.

(vi) The probability of having more than 4 students in the system. *7*

(b) Let $\mathbf{X} \sim N_3(\mu, \Sigma)$, where $\mu = [1 \quad 2 \quad 3]'$ and

$$\Sigma = \begin{bmatrix} 5 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

Find the distribution of

$$\begin{bmatrix} 2X_1 + X_2 + X_3 \\ X_1 - X_2 + X_3 \end{bmatrix}.$$

Also, find the conditional distribution of X_1 given $\begin{bmatrix} X_2 \\ X_3 \end{bmatrix}$. 8

3. (a) Consider the Markov chain with three states, S = {1, 2, 3} following the transition matrix

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \end{array}.$$

(i) Draw the state transition diagram for this chain.

(ii) If $P(X_1 = 1) = P(X_1 = 2) = \frac{1}{4}$, then find $P(X_1 = 3, X_2 = 2, X_3 = 1)$.

(iii) Check whether the chain is irreducible and aperiodic.

(iv) Find the stationary distribution for the chain. 9

(b) Determine the principal components Y_1 and Y_2 for the covariance matrix $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. Also, calculate the proportion of total population variance for the first principal component. 6

4. (a) If $N_1(t)$, $N_2(t)$ are two independent Poisson processes with parameters λ_1 and λ_2 respectively, then show that

$$P[N_1(t) = k \mid N_1(t) + N_2(t) = n] = {}^nC_k \, p^k \, q^{n-k},$$

where $p = \dfrac{\lambda_1}{\lambda_1 + \lambda_2}$, $q = \dfrac{\lambda_2}{\lambda_1 + \lambda_2}$. 4

(b) Let $\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ be a normal random vector with the mean vector $\boldsymbol{\mu} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and covariance matrix $\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$. Suppose $\mathbf{Y} = \mathbf{AX} + \mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \text{ and } \mathbf{Y} \sim N_3.$$

(i) Find $P(0 \leq X_2 \leq 1)$.

(ii) Compute $E(\mathbf{Y})$.

(iii) Find the covariance matrix of $\mathbf{Y}$.

(iv) Find $P(Y_3 \leq 4)$. 7

(c) A box contains two coins : a regular coin and one fake two-headed coin. One coin is chosen at random and tossed twice. The following events are defined :

A : first coin toss results in a head.

B : second coin toss results in a head.

C : coin 1 (regular) has been selected.

Find $P(A \mid C)$, $P(B \mid C)$, $P(A \cap B \mid C)$, $P(A)$, $P(B)$ and $P(A \cap B)$. 4

5. (a) Let the mean vectors and covariance matrices of the variables $\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ and $\mathbf{Y}$ are $\boldsymbol{\mu}_X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\boldsymbol{\mu}_Y = 3$, $\Sigma_{XX} = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$, $\boldsymbol{\sigma}_{YY} = 14$ and $\boldsymbol{\sigma}_{XY} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(i) Fit the equation $\mathbf{Y} = b_0 + b_1 X_1 + b_2 X_2$ as the best linear equation.

(ii) Find the multiple correlation coefficient.

(iii) Find the mean squared error. 6

(b) Suppose the interoccurrence times $\{X_n : n \geq 1\}$ are uniformly distributed on $[0, 1]$. Find the Laplace transform of the renewal function M_t. Also, find $\lim\limits_{t \to \infty} \dfrac{M_t}{t}$. 4

(c) Consider three random variables X_1, X_2, X_3 having the covariance matrix $\begin{bmatrix} 1 & 0{\cdot}12 & 0{\cdot}08 \\ 0{\cdot}12 & 1 & 0{\cdot}06 \\ 0{\cdot}08 & 0{\cdot}06 & 1 \end{bmatrix}$. Write the factor model, if number of variables and number of factors are 3 and 1 respectively. 5

6. (a) Two samples of sizes 40 and 60, respectively, were drawn from two different lots of a certain manufactured component. Two characteristics X_1 and X_2 were measured for the sampled items. The summary statistics of the measurements for lots 1 and 2 is given below :

$$\overline{X}_1 = \begin{bmatrix} 6 \\ 3 \end{bmatrix}, \ \overline{X}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix},$$

$$S_1 = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}, \ S_2 = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

Assume that X_1 and X_2 are in normal distribution and $\Sigma_1 = \Sigma_2$. Test at 99% level of significance whether $\mu_1 = \mu_2$ or not. [You may like to use the values $F_{2,\ 97}\,(0{\cdot}01) = 4{\cdot}86$, $F_{3,\ 97}\,(0{\cdot}01) = 4{\cdot}05$, $F_{4,\ 97}\,(0{\cdot}01) = 3{\cdot}89$] *9*

(b) A particular component in a machine is replaced instantaneously on failure. The successive component lifetimes are uniformly distributed over the interval [2, 5] years. Further, planned replacements take place every 3 years.

Compute

(i) long-term rate of replacements.

(ii) long-term rate of failures. *6*

7. (a) Suppose $n_1 = 20$ and $n_2 = 30$ observations are made on two variables X_1 and X_2, where $X_1 \sim N_2(\mu_1, \Sigma)$ and $X_2 \sim N_2(\mu_2, \Sigma)$. Given is $\mu_1 = [1 \quad 2]'$, $\mu_2 = [-1 \quad 0]'$ and

$$\Sigma = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}'.$$

Considering equal cost and equal prior probabilities, classify the observation $[-1 \quad 1]'$ in one of the two populations. 7

(b) One of the two teller machines handles withdrawals only, while the other handles deposits only in a bank. The service time of both the machines follows exponential distribution with mean service time 3 minutes. The depositors arrive in the bank at the rate of 16 per hour and withdrawers arrive at the rate of 14 per hour in Poisson distribution. Find the average waiting times of depositors and withdrawers in the queue. If each machine can handle both the jobs of deposits and withdrawals, then what will be the average waiting time in the queue for a customer ? 8

8. State whether the following statements are ***true*** or ***false***. Give a short proof or a counter-example in support of your answer. *10*

(i) The matrix $\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$ is a variance-covariance matrix of two-dimensional random variable.

(ii) The quadratic form $x_1^2 - x_2^2$ is positive definite.

(iii) If a subset of state space of a Markov chain is closed, then any state of that subset can communicate with a state outside the state subset.

(iv) Principal components depend on the scales used to measure the variables.

(v) If X and Y are two random variables with $V(X) = V(Y) = 2$, then $-2 < \text{cov}(X, Y) < 2$.